ECEN 615
Methods of Electric Power Systems Analysis

Lecture 14: Power Flow Sensitivities

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Announcements

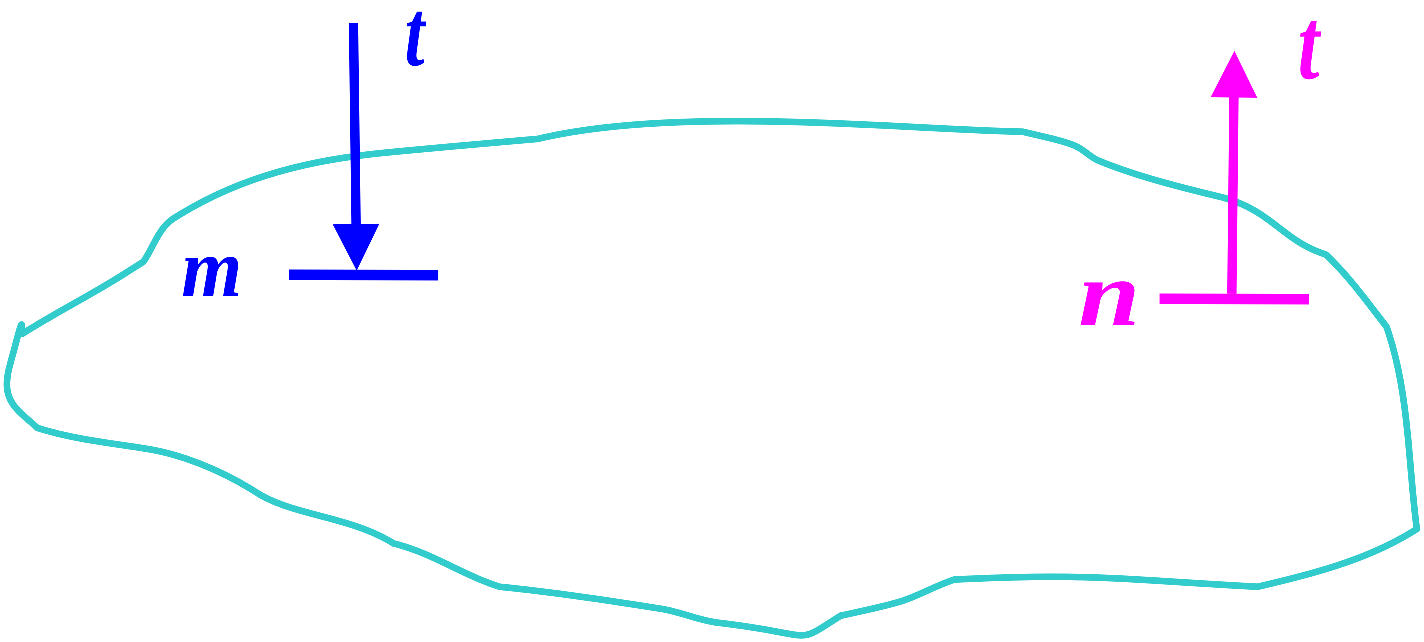
- Starting reading Chapter 9
- Homework 4 is due on Thursday October 14.

Distribution Factors

- Various additional distribution factors may be defined
  - power transfer distribution factor (PTDF)
  - line outage distribution factor (LODF)
  - line addition distribution factor (LADF)
  - outage transfer distribution factor (OTDF)
- These factors may be derived from the ISFs making judicious use of the superposition principle

Definition: Basic Transaction

- A basic transaction involves the transfer of a specified amount of power t from an injection node m to a withdrawal node n



Definition: Basic Transaction

- We use the notation

$$w \; @ \{ m, n, t \}$$

injection node

withdrawal node

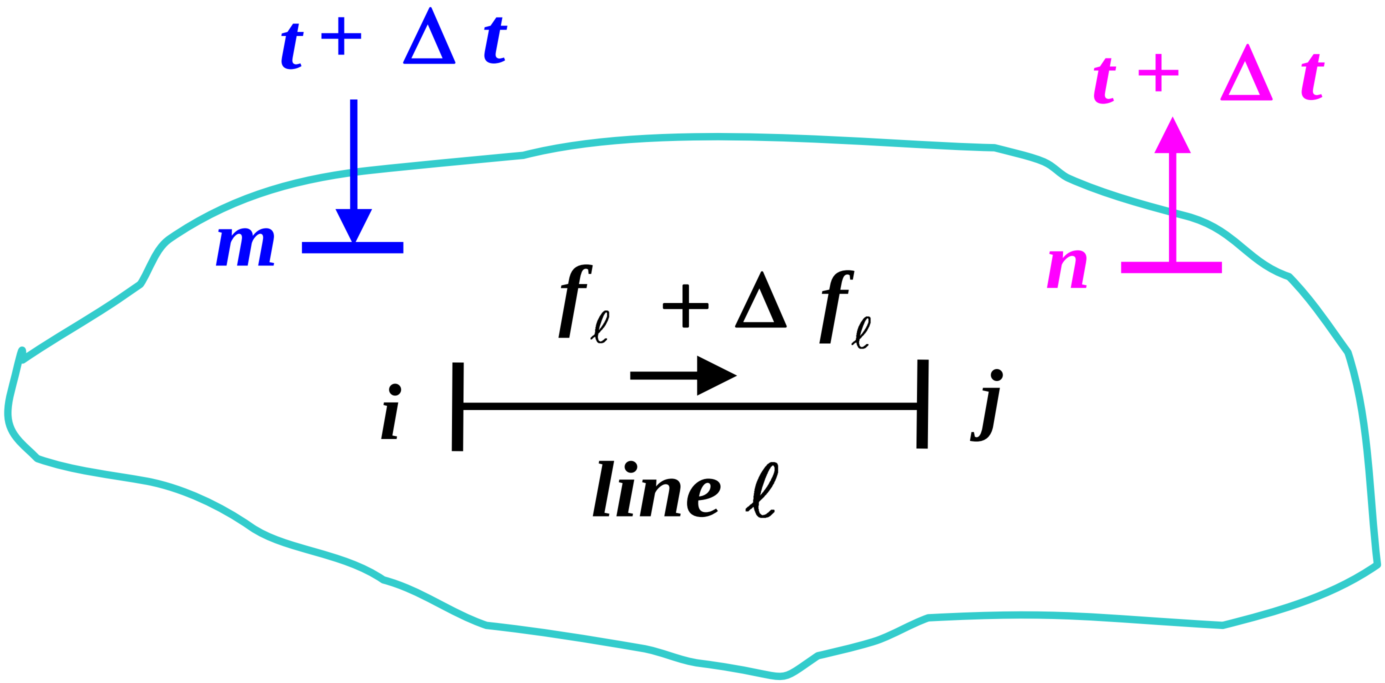
quantity

to denote a basic transaction

Definition: PTDF

- NERC defines a PTDF as
  - "In the pre-contingency configuration of a system under study, a measure of the responsiveness or change in electrical loadings on transmission system Facilities due to a change in electric power transfer from one area to another, expressed in percent (up to 100%) of the change in power transfer"
  - Transaction dependent
- We'll use the notation $\varphi_{\ell}^{(w)}$ to indicate the PTDF on line ℓ with respect to basic transaction *w*
- In the lossless formulation presented here (and commonly used) it is slack bus independent

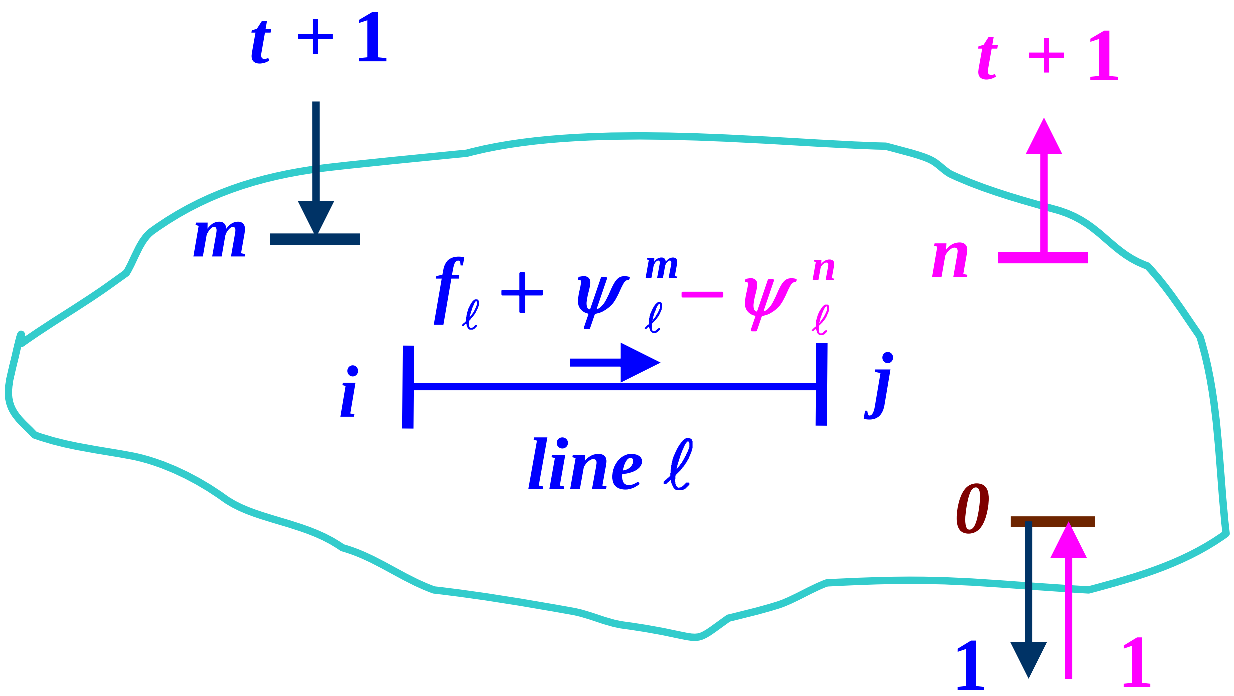
PTDFs



$$\varphi_{\ell}^{(w)} \;@\; \frac{\Delta f_{\ell}}{\Delta t}$$

Note, the PTDF is independent of the amount t; which is often expressed as a percent

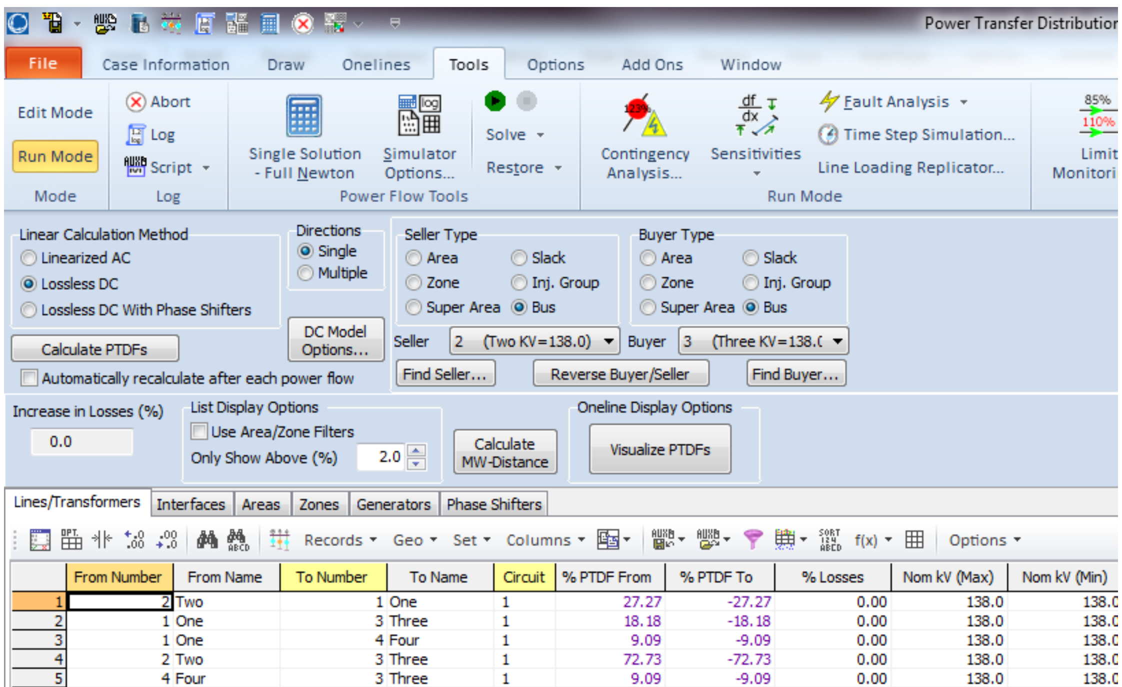
PTDF Evaluation



$$\varphi_{\ell}^{(w)} = \psi_{\ell}^{m} - \psi_{\ell}^{n}$$

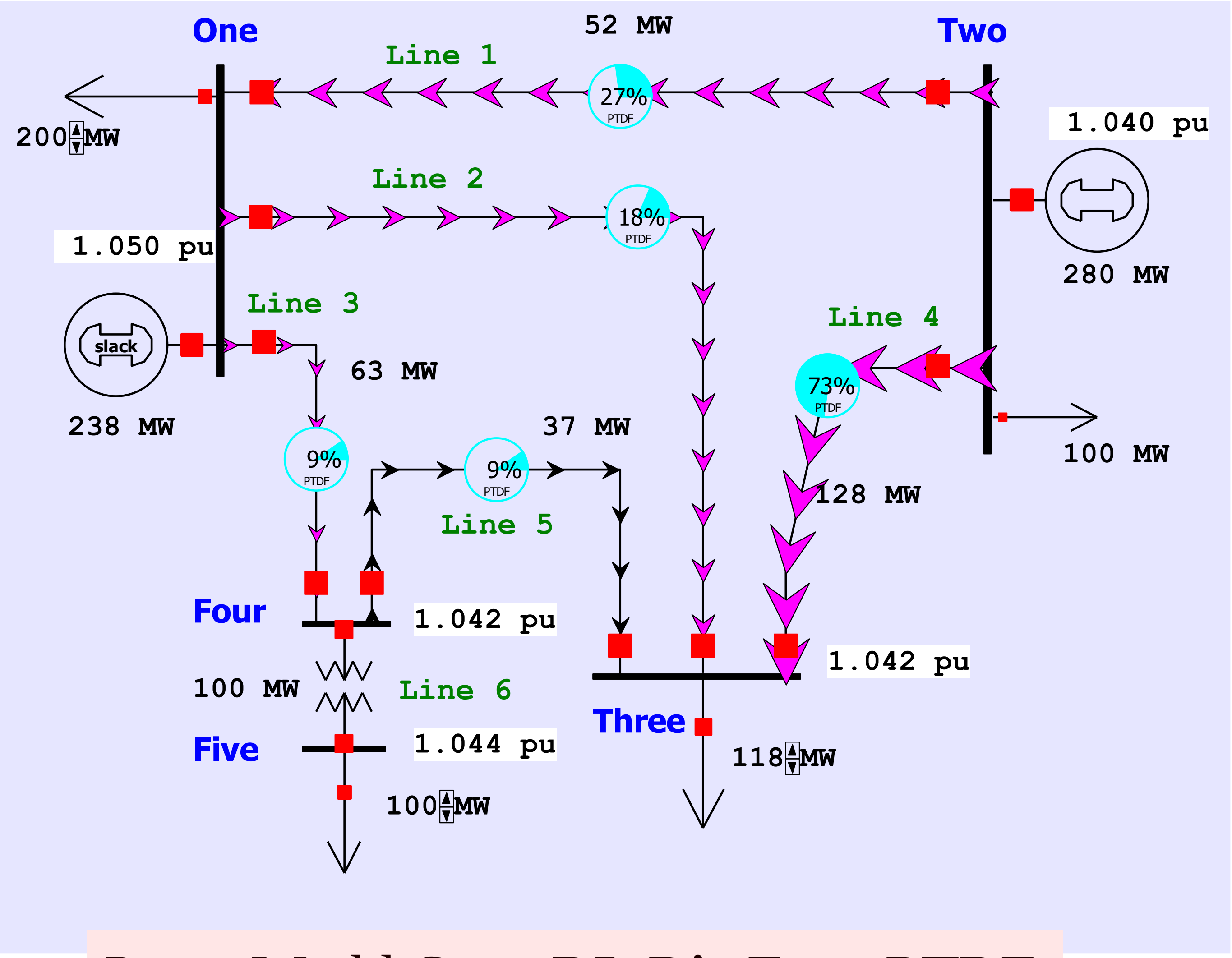
Calculating PTDFs in PowerWorld

- PowerWorld provides a number of options for calculating and visualizing PTDFs
  - Select Tools, Sensitivities, Power Transfer Distribution Factors (PTDFs)



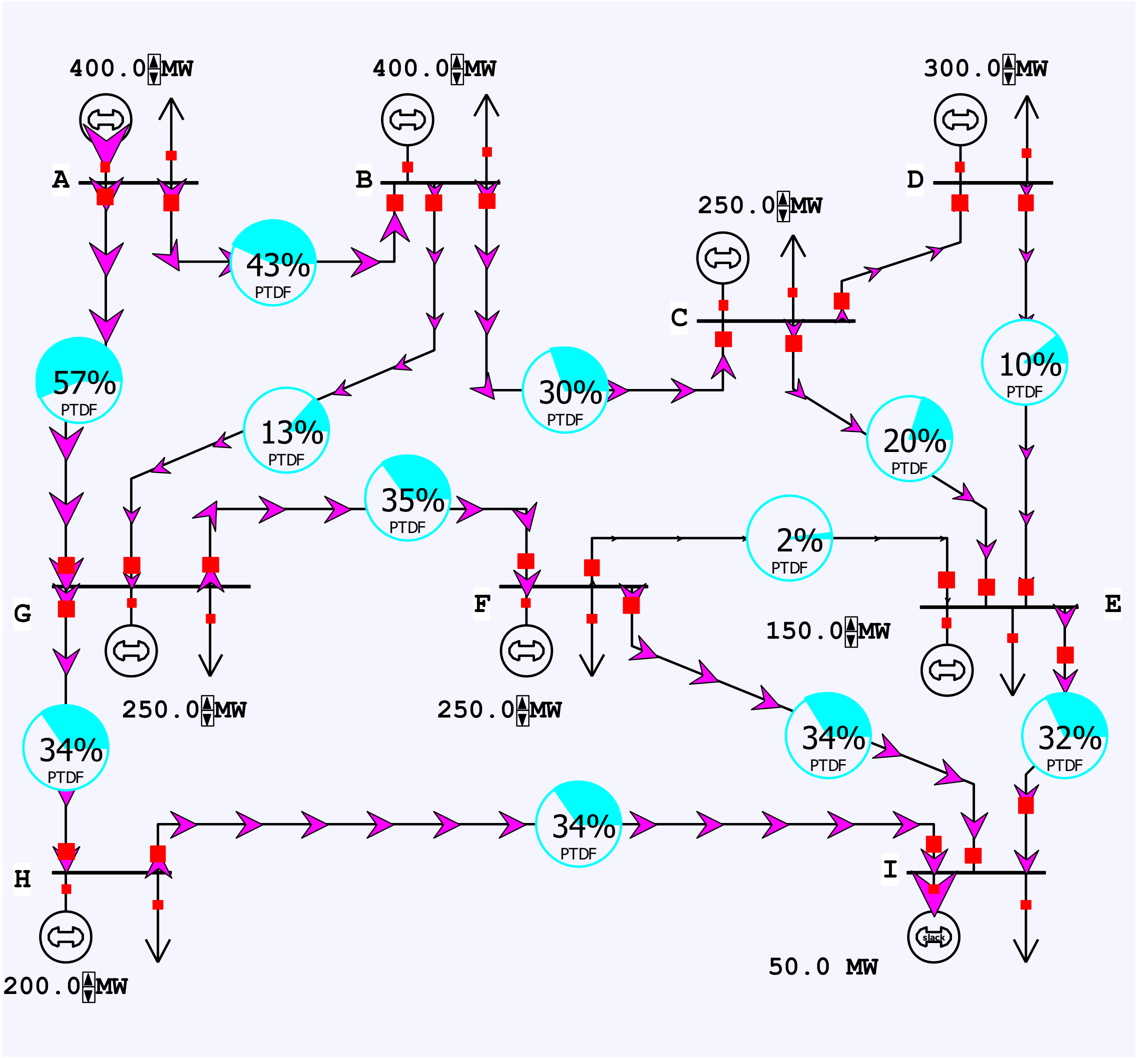
Results are shown for the five bus case for the Bus 2 to Bus 3 transaction

Five Bus PTDF Visualization



PowerWorld Case: **B5_DistFact_PTDF**

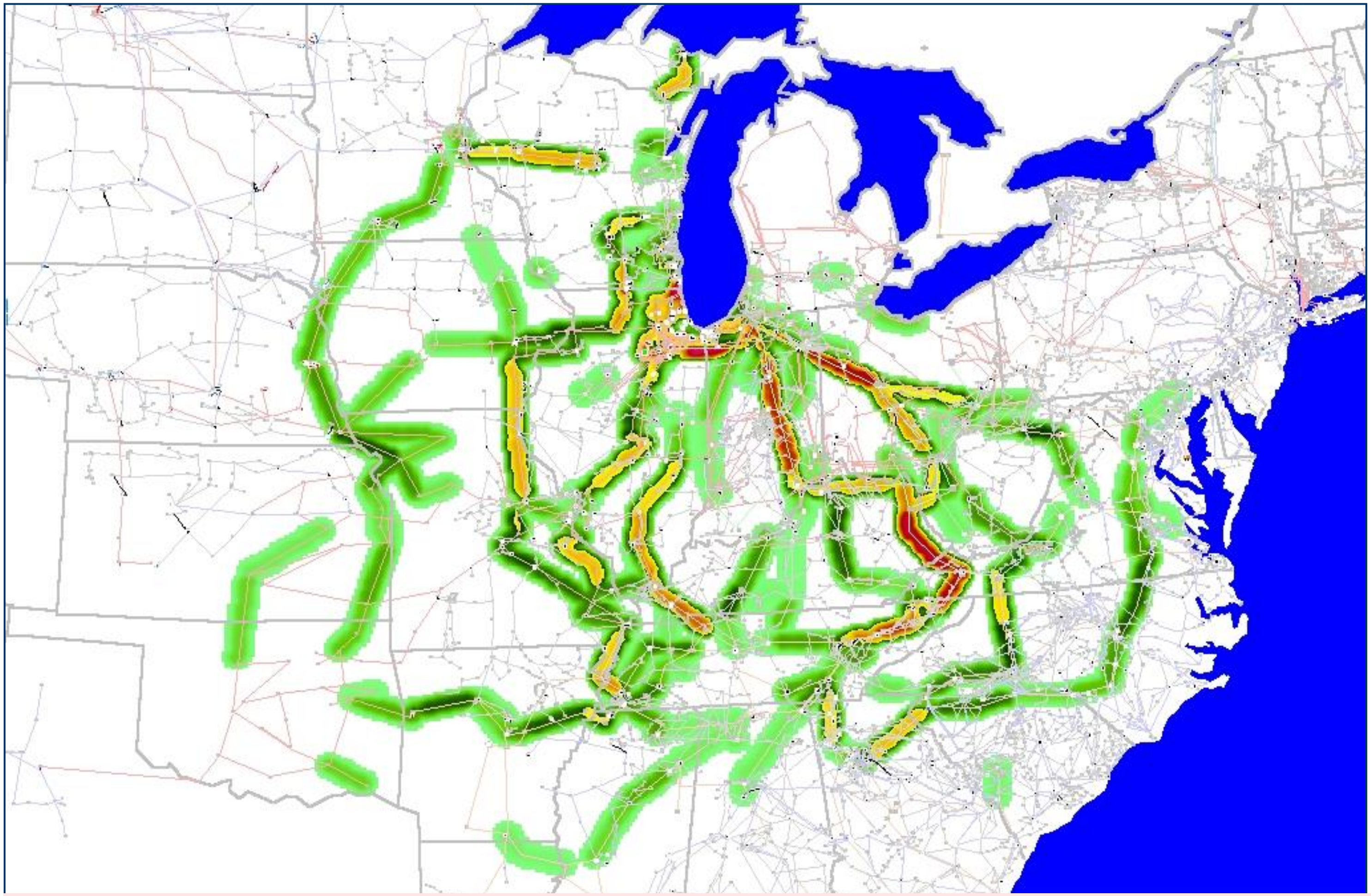
Nine Bus PTDF Example



Display shows the PTDFs for a basic transaction from Bus A to Bus I. Note that 100% of the transaction leaves Bus A and 100% arrives at Bus I

PowerWorld Case: **B9_PTDF**

Eastern Interconnect Example: Wisconsin Utility to TVA PTDFs



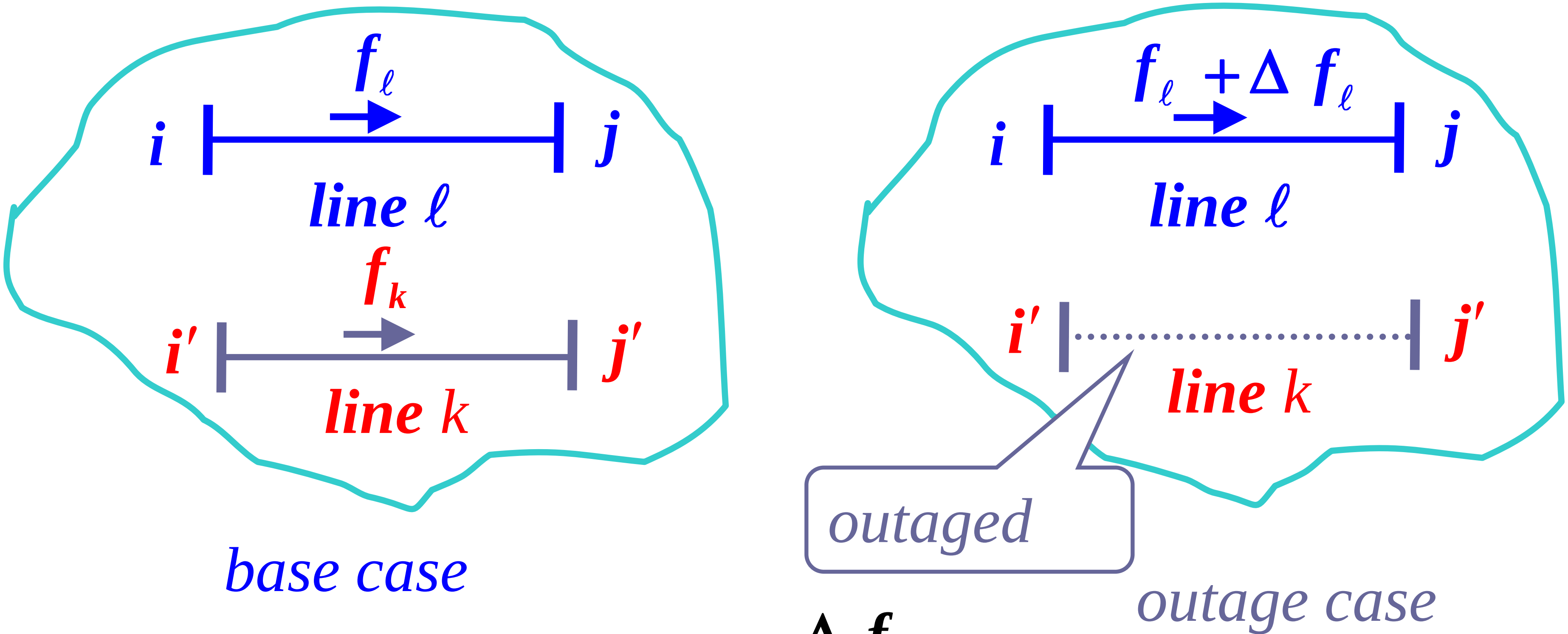
In this example multiple generators contribute for both the seller and the buyer

Contours show lines that would carry at least 2% of a power transfer from Wisconsin to TVA

Line Outage Distribution Factors (LODFs)

- Power system operation is practically always limited by contingencies, with line outages comprising a large number of the contingencies
- Desire is to determine the impact of a line outage (either a transmission line or a transformer) on other system real power flows without having to explicitly solve the power flow for the contingency
- These values are provided by the LODFs
- The LODF d_{ℓ}^{k} is the portion of the pre-outage real power line flow on line k that is redistributed to line ℓ as a result of the outage of line k

LODFs

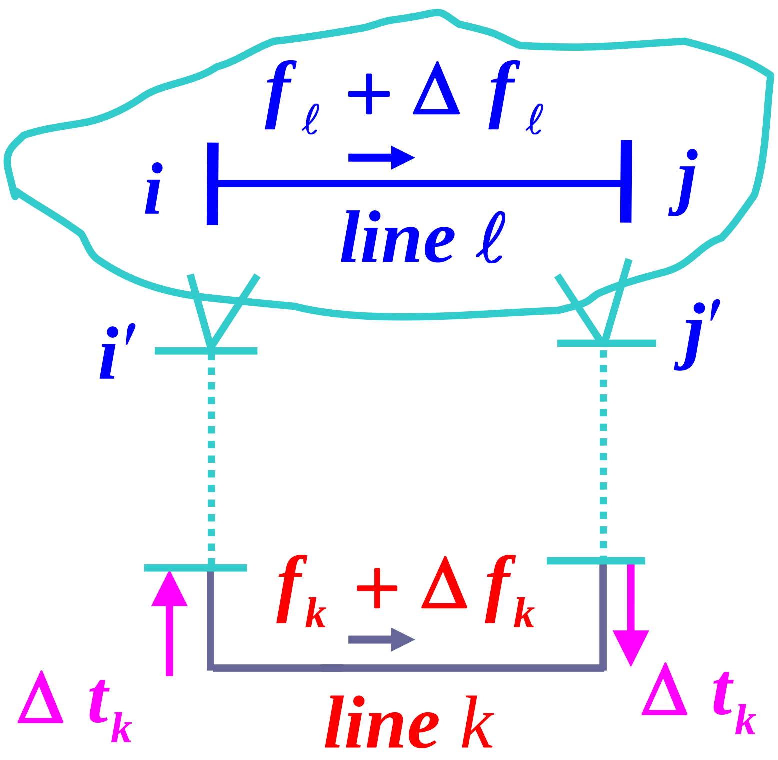


$$d_\ell^k = \frac{\Delta f_\ell}{f_k} = d_{\ell,k}$$

Best reference is Chapter 7 of the course book

LODF Evaluation

We simulate the impact of the outage of line k by adding the basic transaction $w_k = \{ i', j', \Delta t_k \}$



and selecting Δt_k in such a way that the flows on the dashed lines become exactly zero

In general this Δt_k is not equal to the original line flow

LODF Evaluation

- We select Δt_k to be such that

$$f_k + \Delta f_k - \Delta t_k = 0$$

where Δf_k is the active power flow change on the line k due to the transaction w_k

- The line k flow from basic transaction w_k depends on its PTDF $\Delta f_k = \varphi_k^{(w_k)} \Delta t_k$

it follows that $\Delta t_k = \dfrac{f_k}{1 - \varphi_k^{(w_k)}} = \dfrac{f_k}{1 - \left(\psi_k^{i'} - \psi_k^{j'} \right)}$

LODF Evaluation

- For the rest of the network the impacts of the outage of line k are the same as the impacts of the additional basic transaction w_k

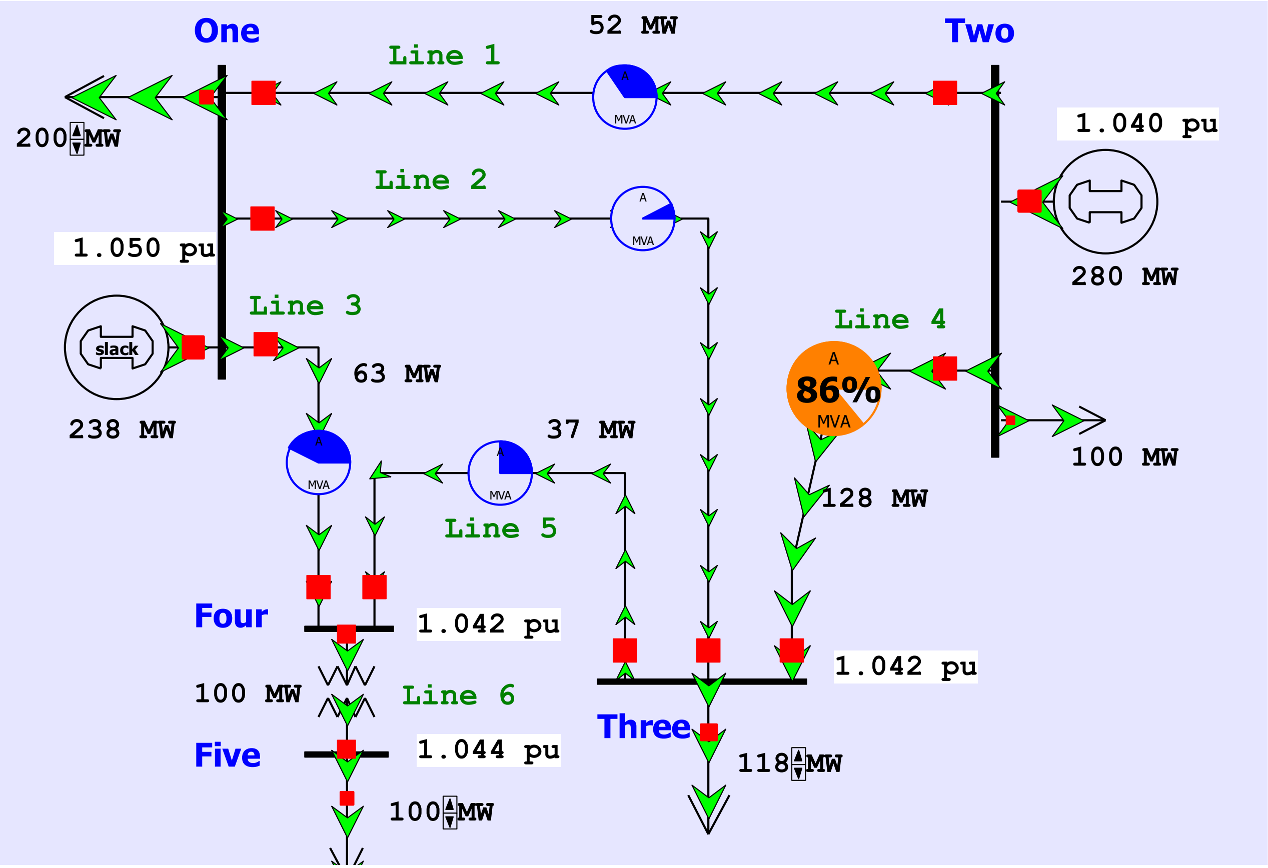
$$\Rightarrow \quad \Delta f_\ell = \varphi_\ell^{(w_k)} \Delta t_k = \frac{\varphi_\ell^{(w_k)}}{1 - \varphi_k^{(w_k)}} f_k$$

- Therefore, by definition the LODF is

$$d_\ell^k = \frac{\Delta f_\ell}{f_k} = \frac{\varphi_\ell^{(w_k)}}{1 - \varphi_k^{(w_k)}}$$

Five Bus Example

- Assume we wish to calculate the values for the outage of line 4 (between buses 2 and 3); this is line k



Say we wish to know the change in flow on the line 3 (Buses 3 to 4). PTDFs for a transaction from 2 to 3 are 0.7273 on line 4 and 0.0909 on line 3

Five Bus Example

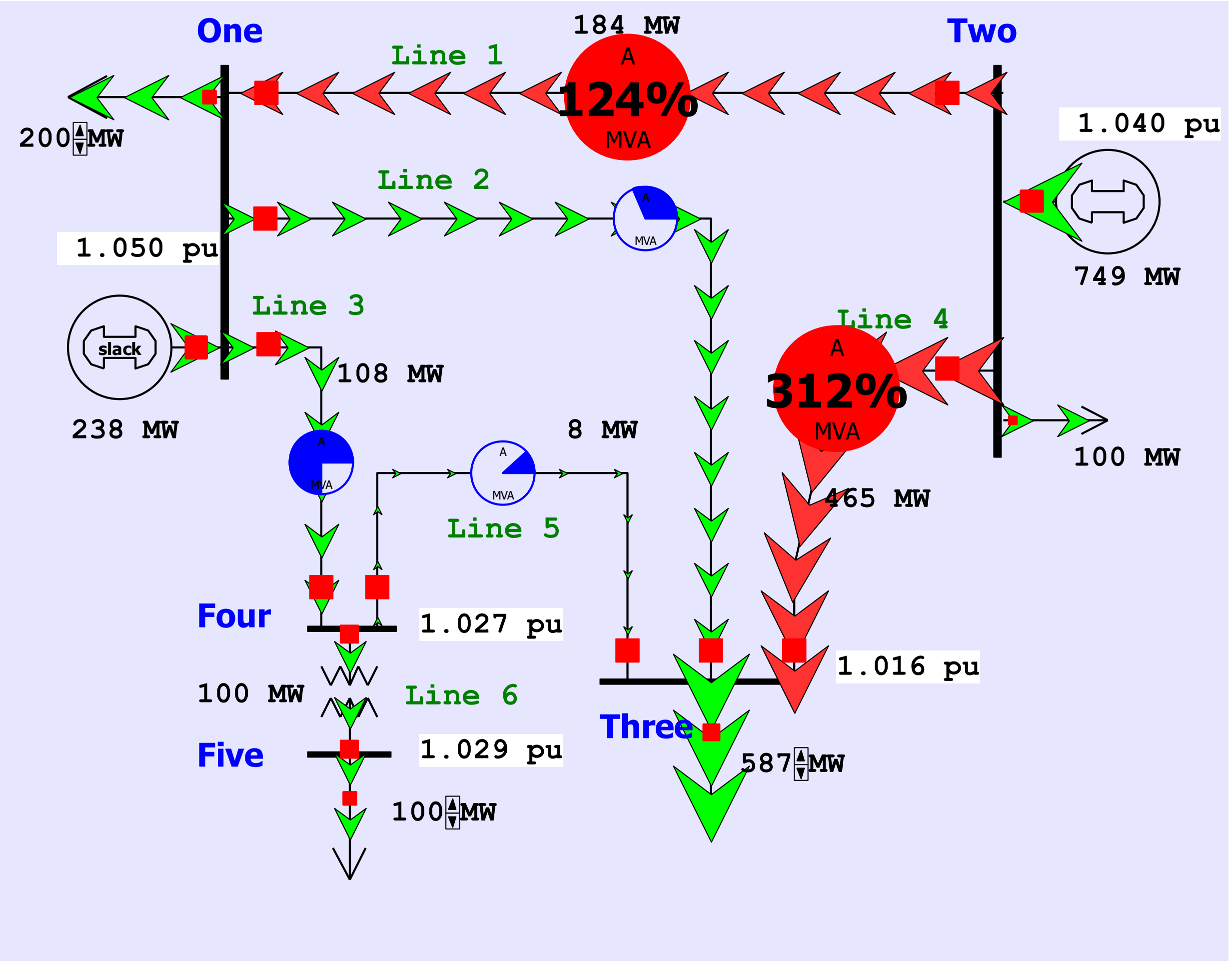
- Hence we get

$$\Delta t_k = \frac{f_k}{1-\varphi_k^{(w_k)}} = \frac{128}{1-0.7273} = 469.4$$

$$d_3^4 = \frac{\Delta f_3}{f_4} = \frac{\varphi_3^{(w_4)}}{1-\varphi_4^{(w_4)}} = \frac{0.0909}{1-0.7273} = 0.333$$

$$\Delta f_3 = (0.333)\, f_4 = 0.333 \times 128 = 42.66\,\text{MW}$$

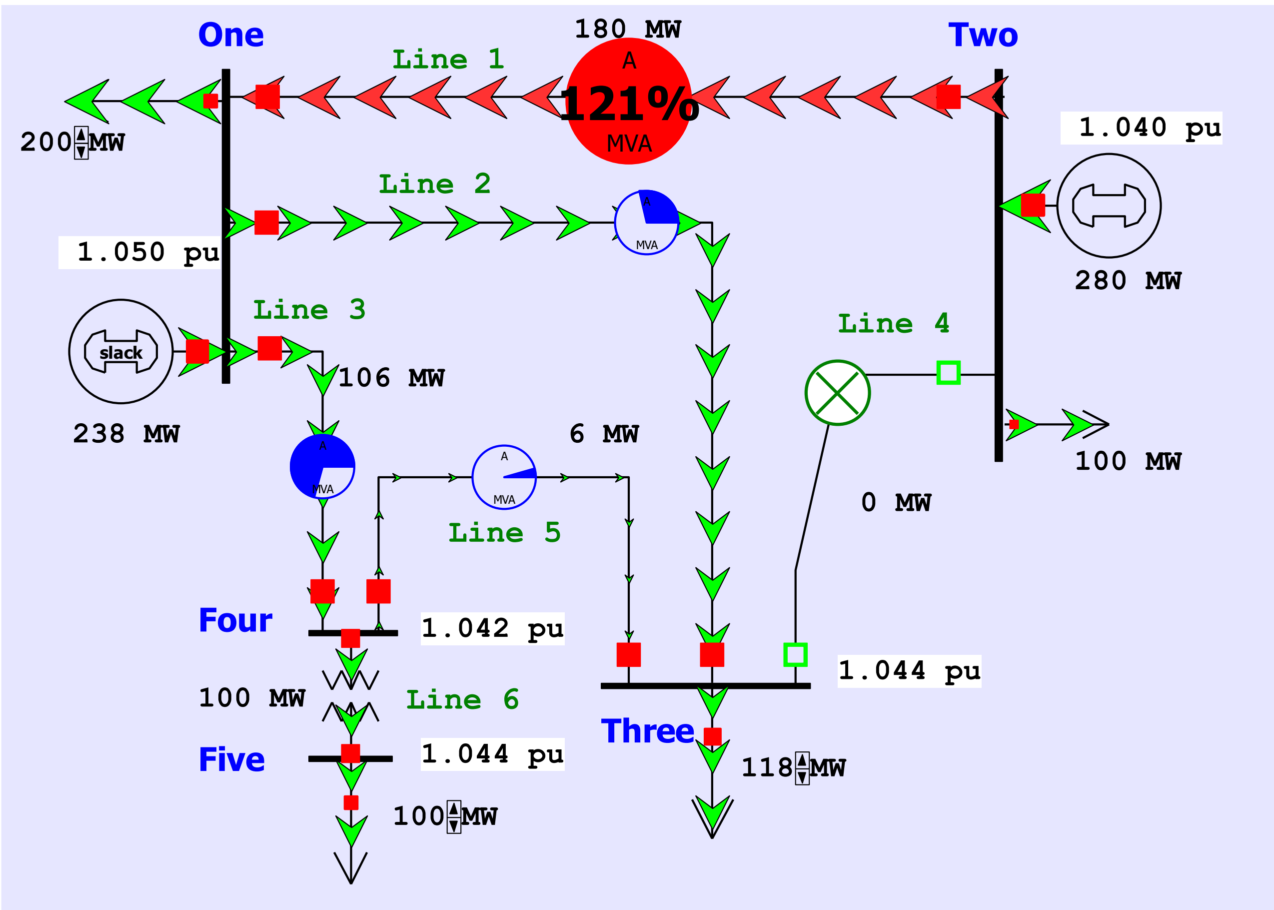
Five Bus Example Compensated



Here is the system with the compensation added to bus 2 and removed at bus 3; we are canceling the impact of the line 4 flow for the reset of the network.

Five Bus Example

- Below we see the network with the line actually outaged



The line 3 flow changed from 63 MW to 106 MW, an increase of 43 MW, matching the LODF value

Developing a Critical Eye

- In looking at the below formula you need to be thinking about what conditions will cause the formula to fail

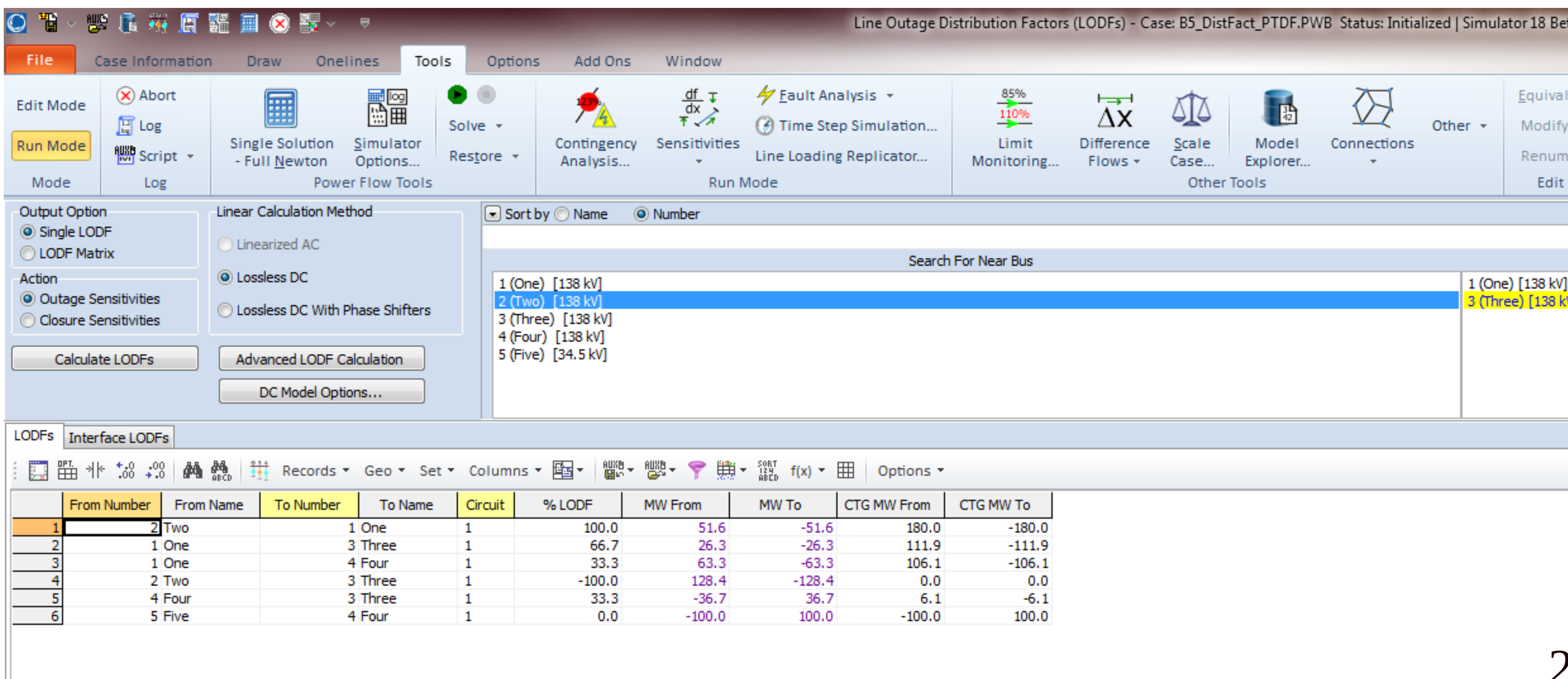
$$\Rightarrow \quad \Delta f_{\ell} = \varphi_{\ell}^{(w_k)} \Delta t_k = \frac{\varphi_{\ell}^{(w_k)}}{1 - \varphi_{k}^{(w_k)}} f_k$$

  Here the obvious situation is when the denominator is zero

- That corresponds to a situation in which the contingency causes system islanding
  - An example is line 6 (between buses 4 and 5)
  - Impact modeled by injections at the buses within each viable island

Calculating LODFs in PowerWorld

- Select Tools, Sensitivities, Line Outage Distribution Factors
  - Select the Line using dialogs on right, and click Calculate LODFS; below example shows values for line 4

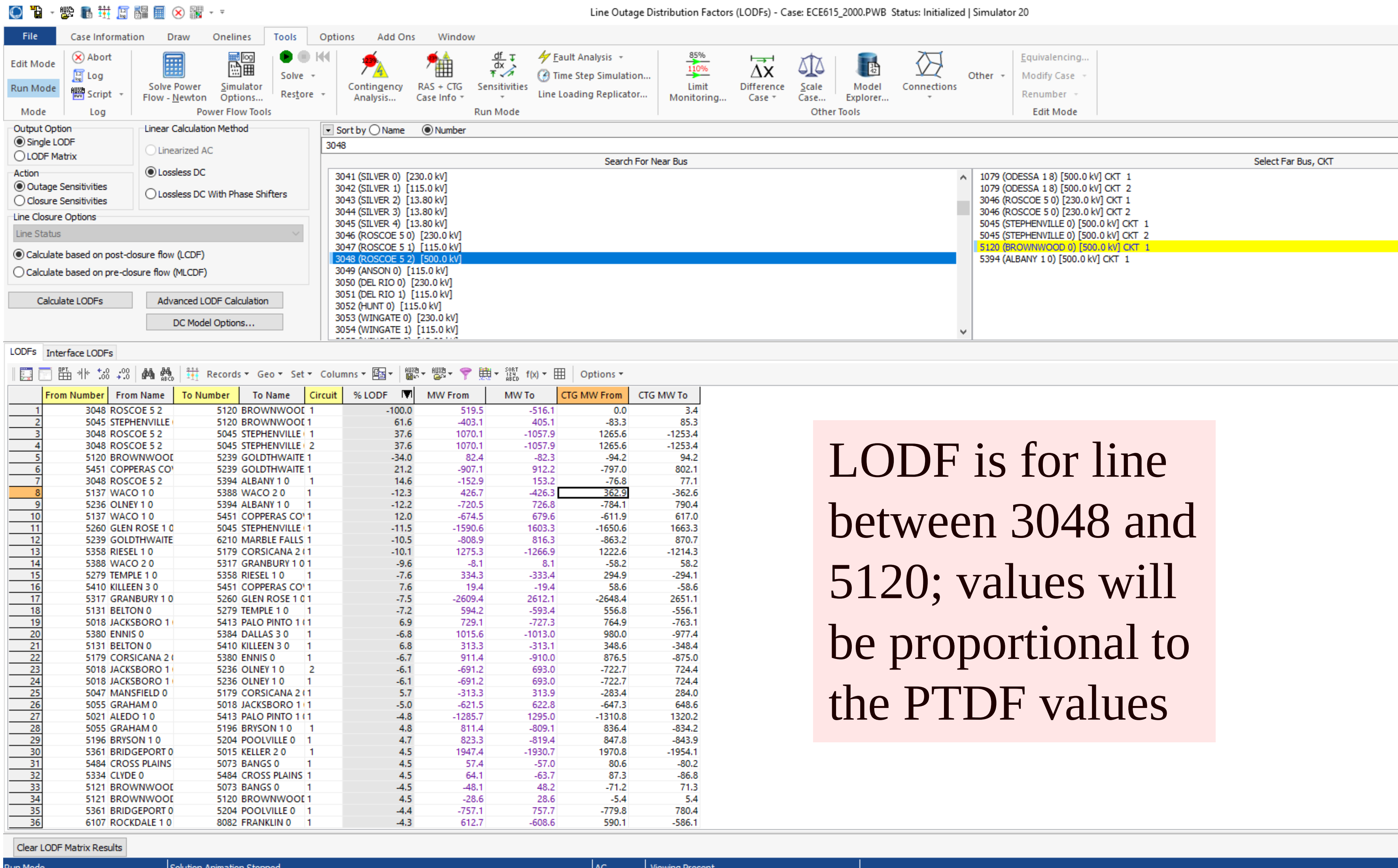


	From Number	From Name	To Number	To Name	Circuit	% LODF	MW From	MW To	CTG MW From	CTG MW To
1	2	Two	1	One	1	100.0	51.6	-51.6	180.0	-180.0
2	1	One	3	Three	1	66.7	26.3	-26.3	111.9	-111.9
3	1	One	4	Four	1	33.3	63.3	-63.3	106.1	-106.1
4	2	Two	3	Three	1	-100.0	128.4	-128.4	0.0	0.0
5	4	Four	3	Three	1	33.3	-36.7	36.7	6.1	-6.1
6	5	Five	4	Four	1	0.0	-100.0	100.0	-100.0	100.0

Blackout Case LODFs

- One of the issues associated with the 8/14/03 blackout was the LODF associated with the loss of the Hanna-Juniper 345 kV line (21350-22163) that was being used in a flow gate calculation was not correct because the Chamberlin-Harding 345 kV line outage was missed
  - With the Chamberlin-Harding line assumed in-service the value was 0.362
  - With this line assumed out-of-service (which indeed it was) the value increased to 0.464

2000 Bus LODF Example



	From Number	From Name	To Number	To Name	Circuit	% LODF	MW From	MW To	CTG MW From	CTG MW To
1	3048	ROSCOE 5 2	5120	BROWNWOOD	1	-100.0	519.5	-516.1	0.0	3.4
2	5045	STEPHENVILLE	5120	BROWNWOOD	1	61.6	-403.1	405.1	-83.3	85.3
3	3048	ROSCOE 5 2	5045	STEPHENVILLE	1	37.6	1070.1	-1057.9	1265.6	-1253.4
4	3048	ROSCOE 5 2	5045	STEPHENVILLE	2	37.6	1070.1	-1057.9	1265.6	-1253.4
5	5120	BROWNWOOD	5239	GOLDTHWAITE	1	-34.0	82.4	-82.3	-94.2	94.2
6	5451	COPPERAS CO	5239	GOLDTHWAITE	1	21.2	-907.1	912.2	-797.0	802.1
7	3048	ROSCOE 5 2	5394	ALBANY 1 0	1	14.6	-152.9	153.2	-76.8	77.1
8	5137	WACO 1 0	5388	WACO 2 0	1	-12.3	426.7	-426.3	362.9	-362.6
9	5236	OLNEY 1 0	5394	ALBANY 1 0	1	-12.2	-720.5	726.8	-784.1	790.4
10	5137	WACO 1 0	5451	COPPERAS CO	1	12.0	-674.5	679.6	-611.9	617.0
11	5260	GLEN ROSE 1 0	5045	STEPHENVILLE	1	-11.5	-1590.6	1603.3	-1650.6	1663.3
12	5239	GOLDTHWAITE	6210	MARBLE FALLS	1	-10.5	-808.9	816.3	-863.2	870.7
13	5358	RIESEL 1 0	5179	CORSICANA 2	1	-10.1	1275.3	-1266.9	1222.6	-1214.3
14	5388	WACO 2 0	5317	GRANBURY 1 0	1	-9.6	-8.1	8.1	-58.2	58.2
15	5279	TEMPLE 1 0	5358	RIESEL 1 0	1	-7.6	334.3	-333.4	294.9	-294.1
16	5410	KILLEEN 3 0	5451	COPPERAS CO	1	7.6	19.4	-19.4	58.6	-58.6
17	5317	GRANBURY 1 0	5260	GLEN ROSE 1 0	1	-7.5	-2609.4	2612.1	-2648.4	2651.1
18	5131	BELTON 0	5279	TEMPLE 1 0	1	-7.2	594.2	-593.4	556.8	-556.1
19	5018	JACKSBORO 1	5413	PALO PINTO 1	1	6.9	729.1	-727.3	764.9	-763.1
20	5380	ENNIS 0	5384	DALLAS 3 0	1	-6.8	1015.6	-1013.0	980.0	-977.4
21	5131	BELTON 0	5410	KILLEEN 3 0	1	6.8	313.3	-313.1	348.6	-348.4
22	5179	CORSICANA 2	5380	ENNIS 0	1	-6.7	911.4	-910.0	876.5	-875.0
23	5018	JACKSBORO 1	5236	OLNEY 1 0	2	-6.1	-691.2	693.0	-722.7	724.4
24	5018	JACKSBORO 1	5236	OLNEY 1 0	1	-6.1	-691.2	693.0	-722.7	724.4
25	5047	MANSFIELD 0	5179	CORSICANA 2	1	5.7	-313.3	313.9	-283.4	284.0
26	5055	GRAHAM 0	5018	JACKSBORO 1	1	-5.0	-621.5	622.8	-647.3	648.6
27	5021	ALEDO 1 0	5413	PALO PINTO 1	1	-4.8	-1285.7	1295.0	-1310.8	1320.2
28	5055	GRAHAM 0	5196	BRYSON 1 0	1	4.8	811.4	-809.1	836.4	-834.2
29	5196	BRYSON 1 0	5204	POOLVILLE 0	1	4.7	823.3	-819.4	847.8	-843.9
30	5361	BRIDGEPORT 0	5015	KELLER 2 0	1	4.5	1947.4	-1930.7	1970.8	-1954.1
31	5484	CROSS PLAINS	5073	BANGS 0	1	4.5	57.4	-57.0	80.6	-80.2
32	5334	CLYDE 0	5484	CROSS PLAINS	1	4.5	64.1	-63.7	87.3	-86.8
33	5121	BROWNWOOD	5073	BANGS 0	1	-4.5	-48.1	48.2	-71.2	71.3
34	5121	BROWNWOOD	5120	BROWNWOOD	1	4.5	-28.6	28.6	-5.4	5.4
35	5361	BRIDGEPORT 0	5204	POOLVILLE 0	1	-4.4	-757.1	757.7	-779.8	780.4
36	6107	ROCKDALE 1 0	8082	FRANKLIN 0	1	-4.3	612.7	-608.6	590.1	-586.1

LODF is for line between 3048 and 5120; values will be proportional to the PTDF values

2000 Bus LODF Example

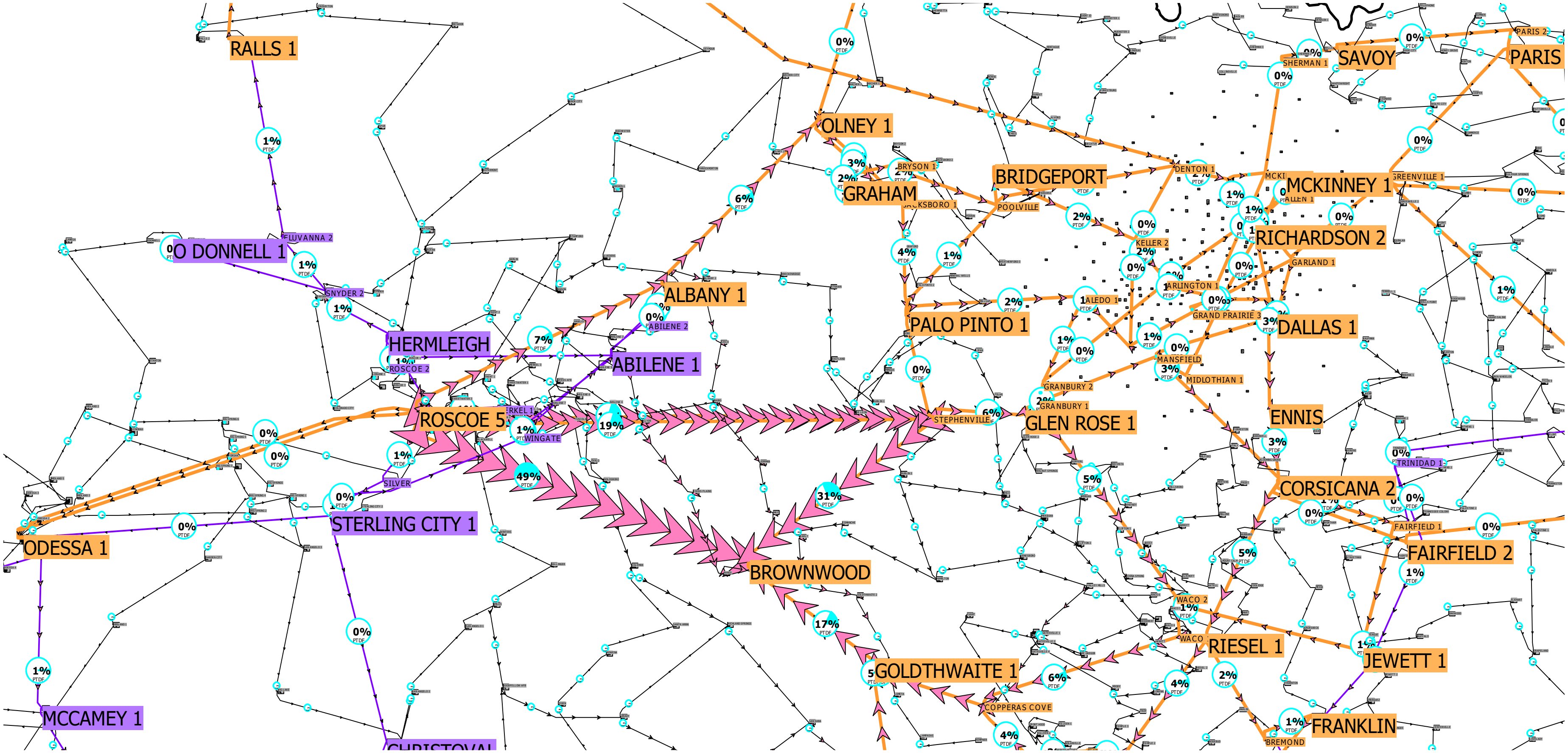


Image visualizes the PTDFs between buses 3048 and 5120

Multiple Line LODFs

- LODFs can also be used to represent multiple device contingencies, but it is usually more involved than just adding the effects of the single device LODFs
- Assume a simultaneous outage of lines k_1 and k_2
- Now setup two transactions, w_{k1} (with value Δt_{k1})and w_{k2} (with value Δt_{k2}) so

$$f_{k1} + \Delta f_{k1} + \Delta f_{k2} - \Delta t_{k1} = 0$$

$$f_{k2} + \Delta f_{k1} + \Delta f_{k2} - \Delta t_{k2} = 0$$

$$f_{k1} + \varphi_{k1}^{(w_{k1})} \Delta t_{k1} + \varphi_{k1}^{(w_{k2})} \Delta t_{k2} - \Delta t_{k1} = 0$$

$$f_{k2} + \varphi_{k2}^{(w_{k1})} \Delta t_{k1} + \varphi_{k2}^{(w_{k2})} \Delta t_{k1} - \Delta t_{k2} = 0$$

Multiple Line LODFs

- Hence we can calculate the simultaneous impact of multiple outages; details for the derivation are given in C.Davis, T.J. Overbye, "Linear Analysis of Multiple Outage Interaction," *Proc. 42nd HICSS*, 2009
- Equation for the change in flow on line ℓ for the outage of lines k_1 and k_2 is

$$\Delta f_{\ell} = \begin{bmatrix} d_{\ell}^{k1} & d_{\ell}^{k2} \end{bmatrix} \begin{bmatrix} 1 & -d_{k1}^{k2} \\ -d_{k2}^{k1} & 1 \end{bmatrix}^{-1} \begin{bmatrix} f_{k1} \\ f_{k2} \end{bmatrix}$$

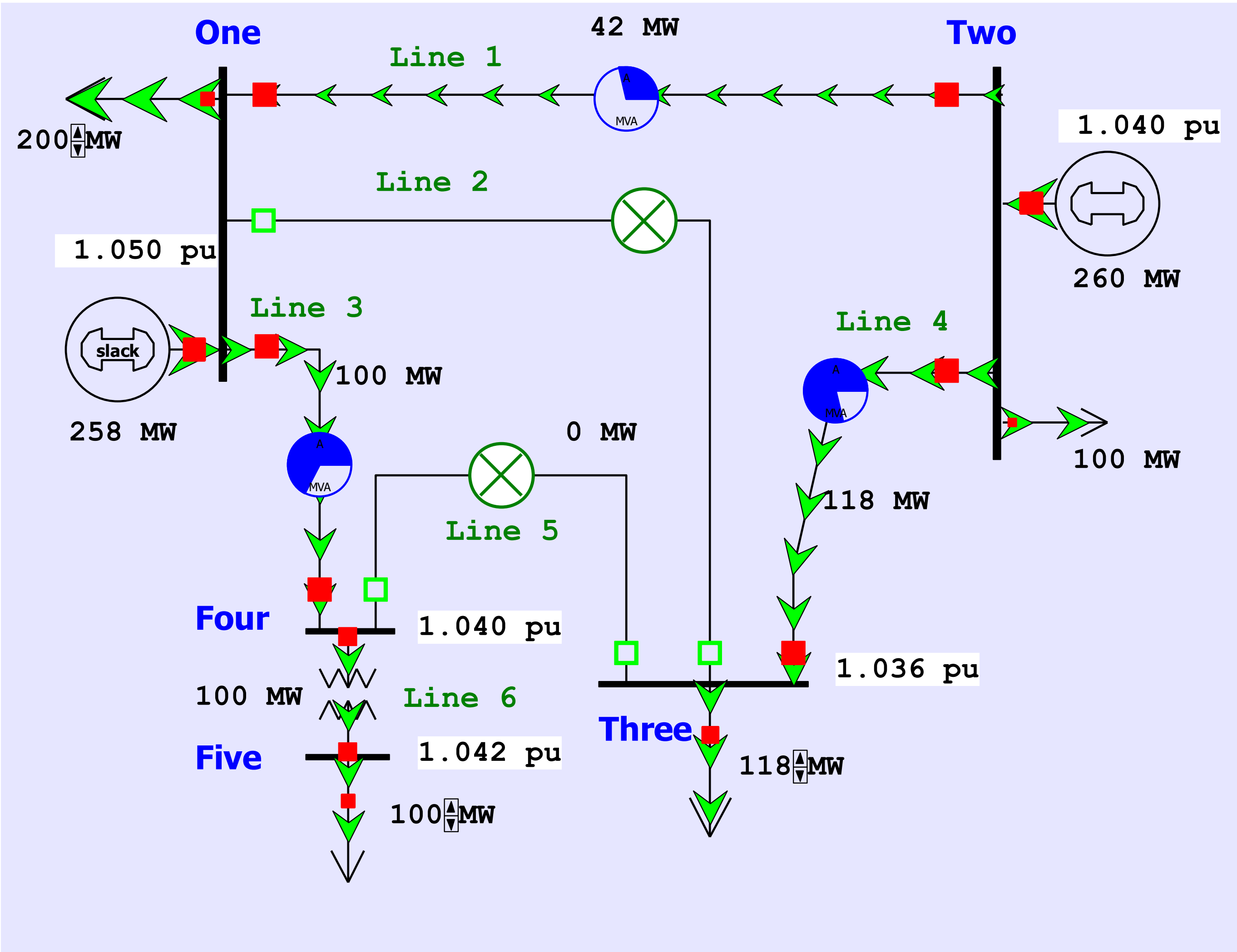
Multiple Line LODFs

- Example: Five bus case, outage of lines 2 and 5 to flow on line 4.

$$\Delta f_\ell = \begin{bmatrix} d_\ell^{k1} & d_\ell^{k2} \end{bmatrix} \begin{bmatrix} 1 & -d_{k1}^{k2} \\ -d_{k2}^{k1} & 1 \end{bmatrix}^{-1} \begin{bmatrix} f_{k1} \\ f_{k2} \end{bmatrix}$$

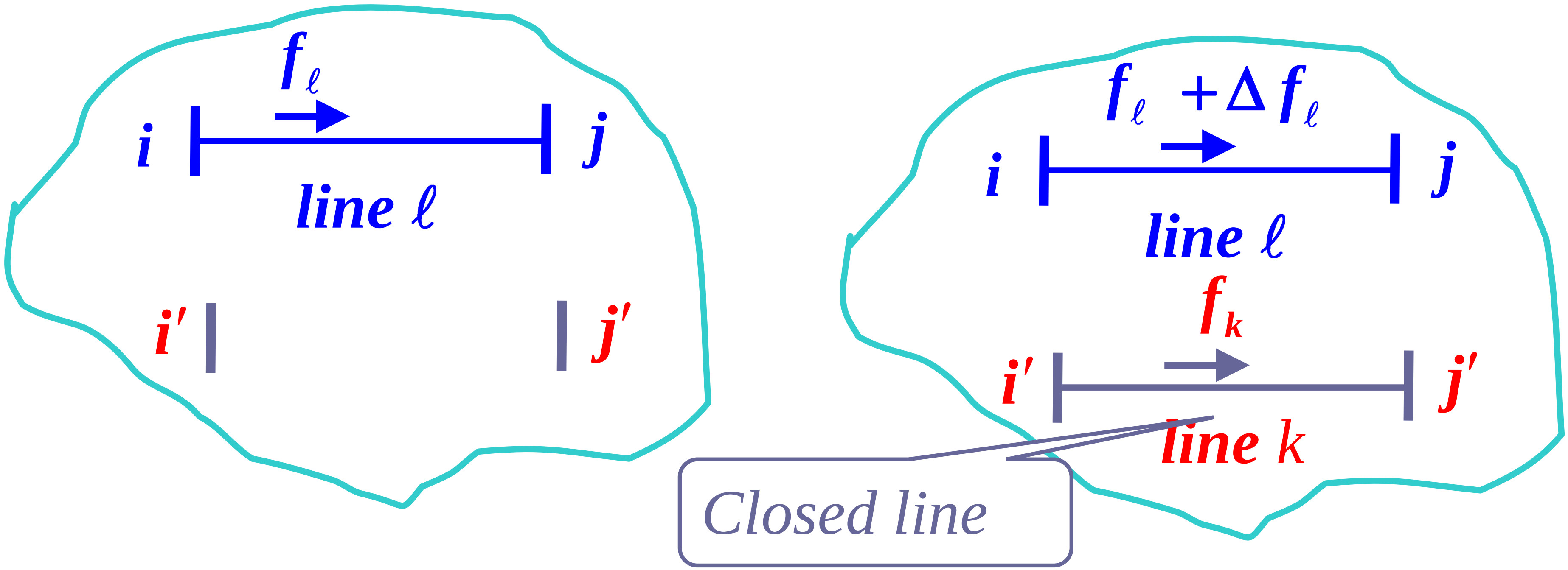
$$\Delta f_\ell = \begin{bmatrix} 0.4 & 0.25 \end{bmatrix} \begin{bmatrix} 1 & -0.75 \\ -0.6 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.336 \\ -0.331 \end{bmatrix} = 0.005$$

Multiple Line LODFs



Flow goes from 117.5 to 118.0

Line Closure Distribution Factors (LCDFs)



base case

line k addition case

$$LCDF_\ell^k = \frac{\Delta f_\ell}{f_k} = LCDF_{\ell,k}$$

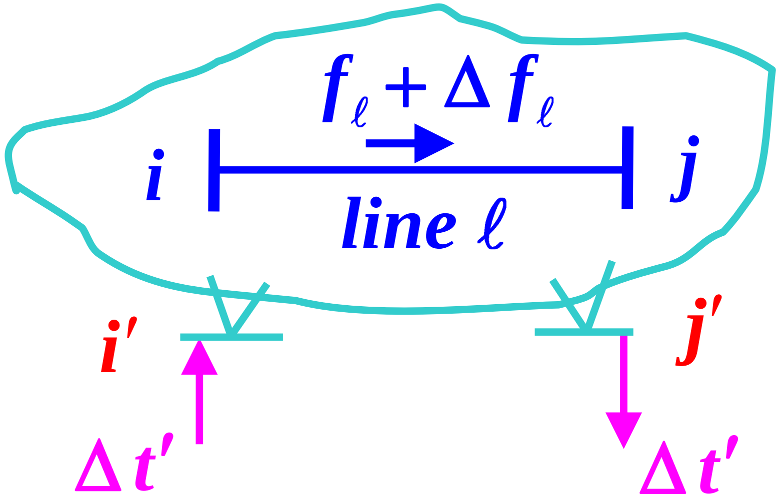
LCDF Definition

- The line closure distribution factor (LCDF), $LCDF_{\ell,k}$, for the closure of line k (or its addition if it does not already exist) is the portion of the line active power flow on line k that is distributed to line ℓ due to the closure of line k
- Since line k is currently open, the obvious question is, "what flow on line k?"
- Answer (in a dc power flow sense) is the flow that will occur when the line is closed (which we do not know)

LCDF Evaluation

- We simulate the impact of the closure of line k by imposing the additional basic transaction

$$w_k = \{ i', j', \Delta t_k \}$$



on the base case network and we select Δt_k so that

$$\Delta t_k = - f_k$$

LCDF Evaluation

- For the other parts of the network, the impacts of the addition of line k are the same as the impacts of adding the basic transaction w_k

$$\Delta f_\ell = \varphi_\ell^{(w_k)} \Delta t_k = -\varphi_\ell^{(w_k)} f_k$$

- Therefore, the definition is

$$LCDF_{\ell,k} = \frac{\Delta f_\ell}{f_k} = -\varphi_\ell^{(w_k)}$$

- The post-closure flow f_k is determined (in a dc power flow sense) as the flow that would occur from the angle difference divided by (1 + $\varphi_k^{(w_k)}$)

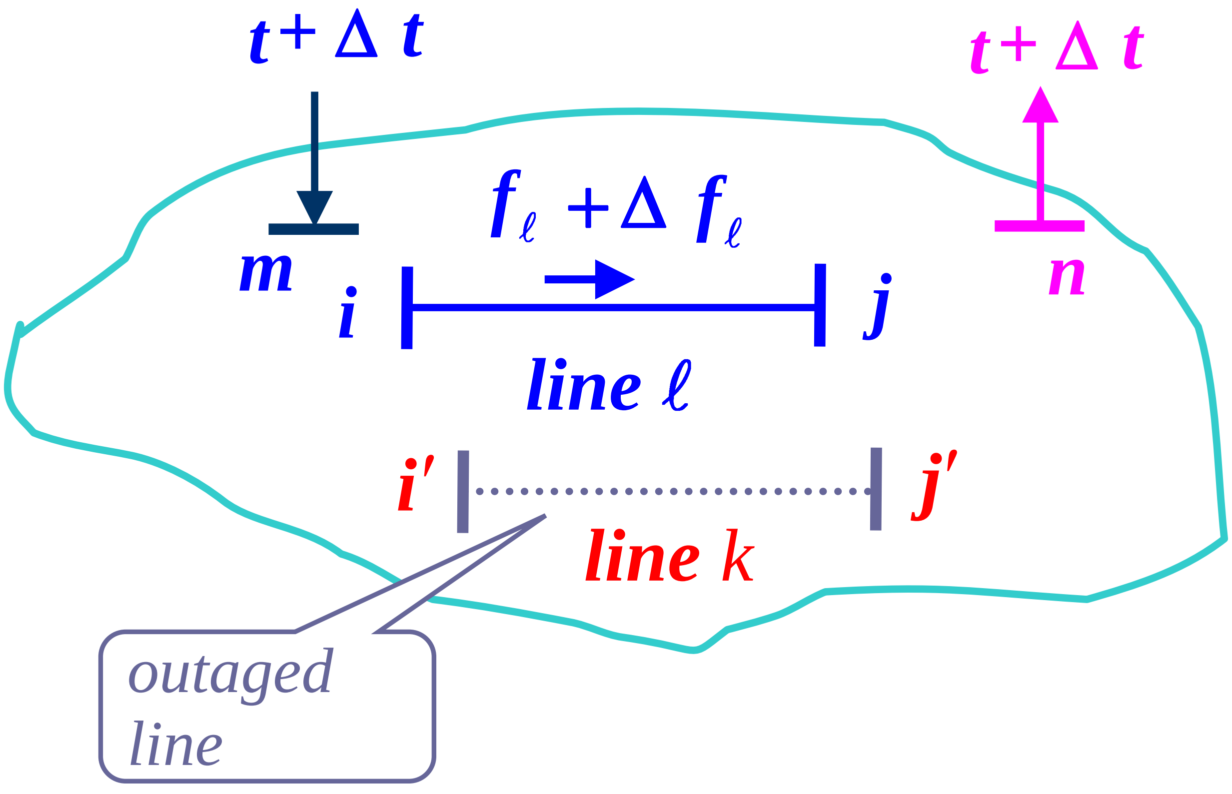
Outage Transfer Distribution Factor

- The outage transfer distribution factor (OTDF) is defined as the PTDF with the line k outaged
- The OTDF applies only to the post-contingency configuration of the system since its evaluation explicitly considers the line k outage

$$\left(\varphi_{\ell}^{(w)} \right)^{k}$$

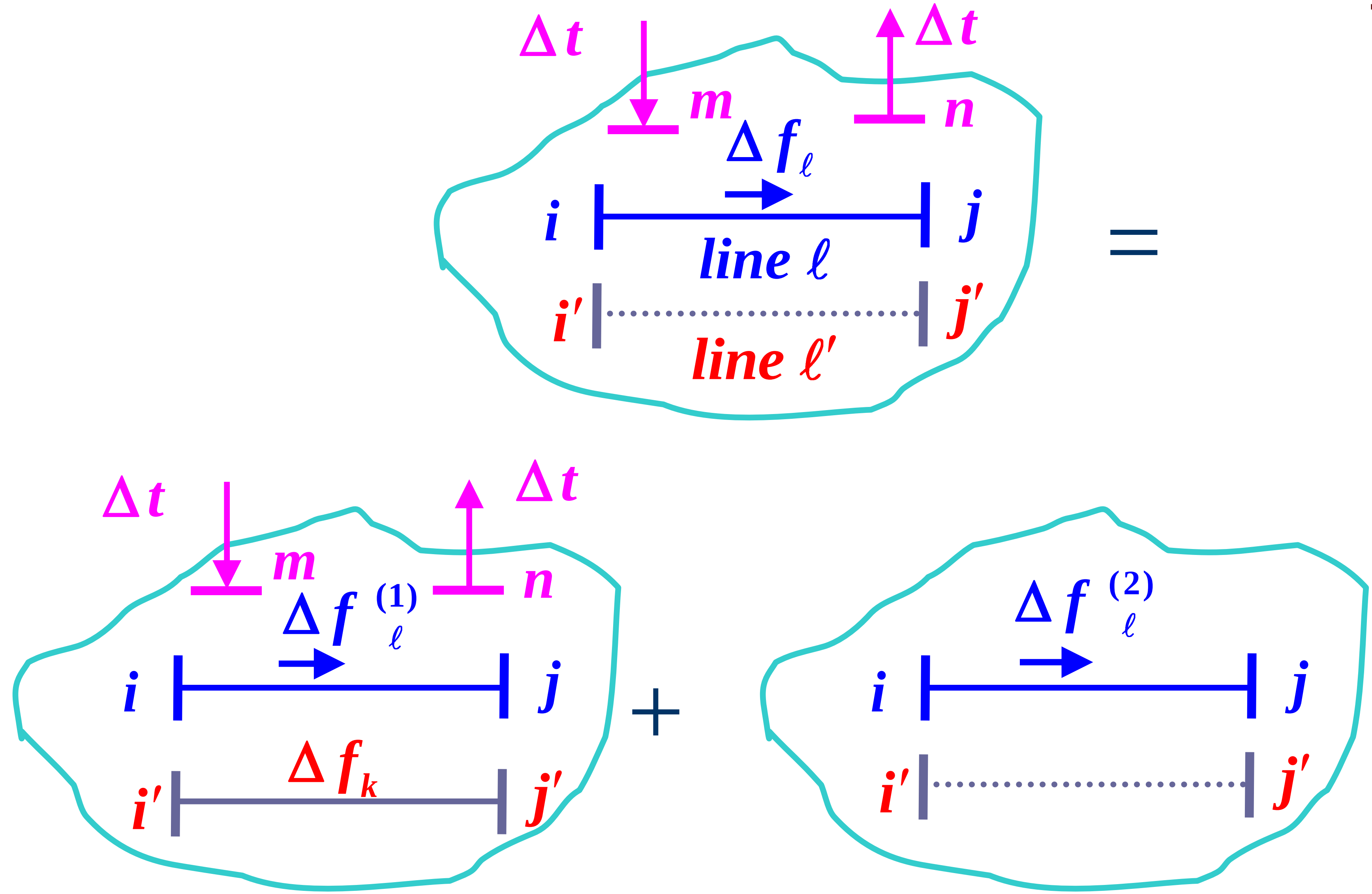
- This is a quite important value since power system operation is usually contingency constrained

Outage Transfer Distribution Factor (OTDF)



$$\left(\varphi_{\ell}^{(w)}\right)^{k} @ \left.\frac{\Delta f_{\ell}}{\Delta t}\right|_{k\ outaged}$$

OTDF Evaluation



OTDF Evaluation

- Since $\Delta f_{\ell}^{(1)} = \varphi_{\ell}^{(w)} \Delta t$

  and $\Delta f_k = \varphi_k^{(w)} \Delta t$

  then $\Delta f_{\ell}^{(2)} = d_{\ell}^{k} \Delta f_k = d_{\ell}^{k} \varphi_k^{(w)} \Delta t$

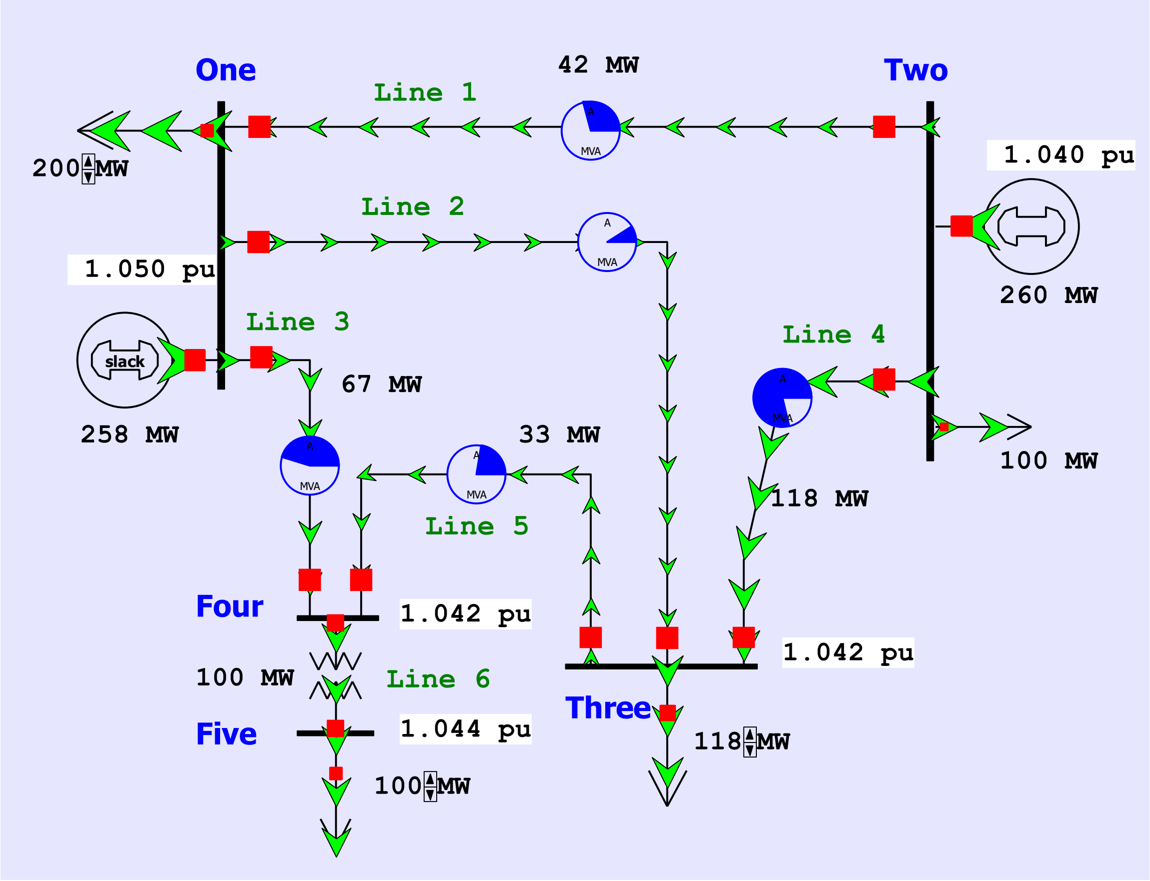
  so that

$$\Delta f_{\ell} = \Delta f_{\ell}^{(1)} + \Delta f_{\ell}^{(2)} = \left[\varphi_{\ell}^{(w)} + d_{\ell}^{k} \varphi_k^{(w)} \right] \Delta t$$

$$\left(\varphi_{\ell}^{(w)} \right)^{k} = \varphi_{\ell}^{(w)} + d_{\ell}^{k} \varphi_k^{(w)}$$

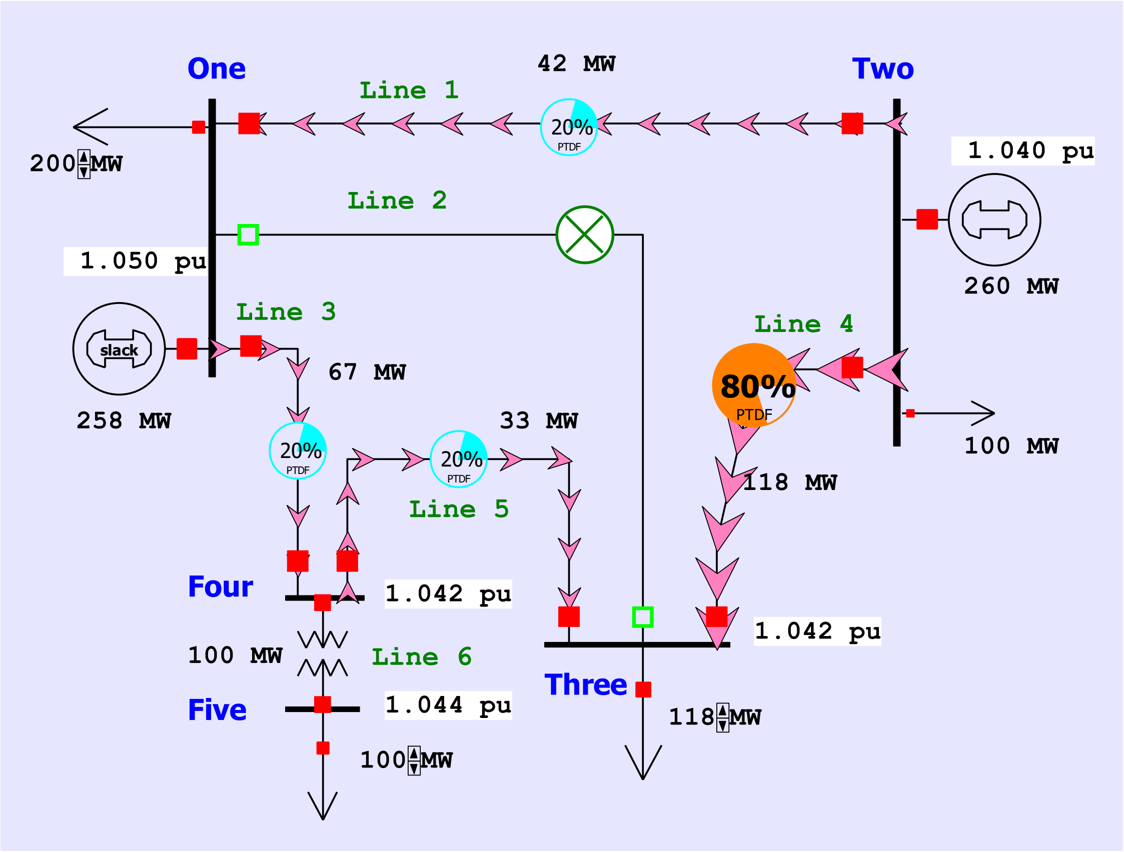
Five Bus Example

- Say we would like to know the PTDF on line 1 for a transaction between buses 2 and 3 with line 2 out



Five Bus Example

- Hence we want to calculate these values without having to explicitly outage line 2



Hence the value we are looking for is 0.2 (20%)

Five Bus Example

- Evaluating: the PTDF for the bus 2 to 3 transaction on line 1 is 0.2727; it is 0.1818 on line 2 (from buses 1 to 3); the LODF is on line 1 for the outage of line 2 is -0.4
- Hence $\left(\varphi_{\ell}^{(w)}\right)^{k} = \varphi_{\ell}^{(w)} + d_{\ell}^{k}\varphi_{k}^{(w)}$

$$\mathbf{0.2727 + (-0.4) \times (0.1818) = 0.200}$$

- For line 4 (buses 2 to 3) the value is

$$\mathbf{0.7273 + (0.4) \times (0.1818) = 0.800}$$

August 14, 2003 OTDF Example

- Flowgate 2264 monitored the flow on Star-Juniper 345 kV line for contingent loss of Hanna-Juniper 345 kV normally the LODF for this flowgate is 0.361
  - flowgate had a limit of 1080 MW
  - at 15:05 EDT the flow as 517 MW on Star-Juniper, 1004 MW on Hanna-Juniper, giving a flowgate value of 520+0.361*1007=884 (82%)
  - Chamberlin-Harding 345 opened at 15:05, but was missed
  - At 15:06 EDT (after loss of Chamberlin-Harding 345) #2265 had an incorrect value because its LODF was not updated.
  - Value should be 633+0.463*1174=1176 (109%)
  - Value was 633 + 0.361*1174=1057 (98%)

UTC Revisited

- We can now revisit the uncommitted transfer capability (UTC) calculation using PTDFs and LODFs
- Recall trying to determine maximum transfer between two areas (or buses in our example)
- For base case maximums are quickly determined with PTDFs

$$u_{m,n}^{(0)} = \min_{\varphi_{\ell}^{(w)} > 0} \left\{ \frac{f_{\ell}^{max} - f_{\ell}^{(0)}}{\varphi_{\ell}^{(w)}} \right\}$$

Note we are ignoring zero (or small) PTDFs; would also need to consider flow reversal

UTC Revisited

- For the contingencies we use

$$u_{m,n}^{(1)} = \min_{\left(\varphi_\ell^{(w)}\right)^k > 0} \left\{ \frac{f_\ell^{max} - f_\ell^{(0)} - d_\ell^k f_k^{(0)}}{\left(\varphi_\ell^{(w)}\right)^k} \right\}$$

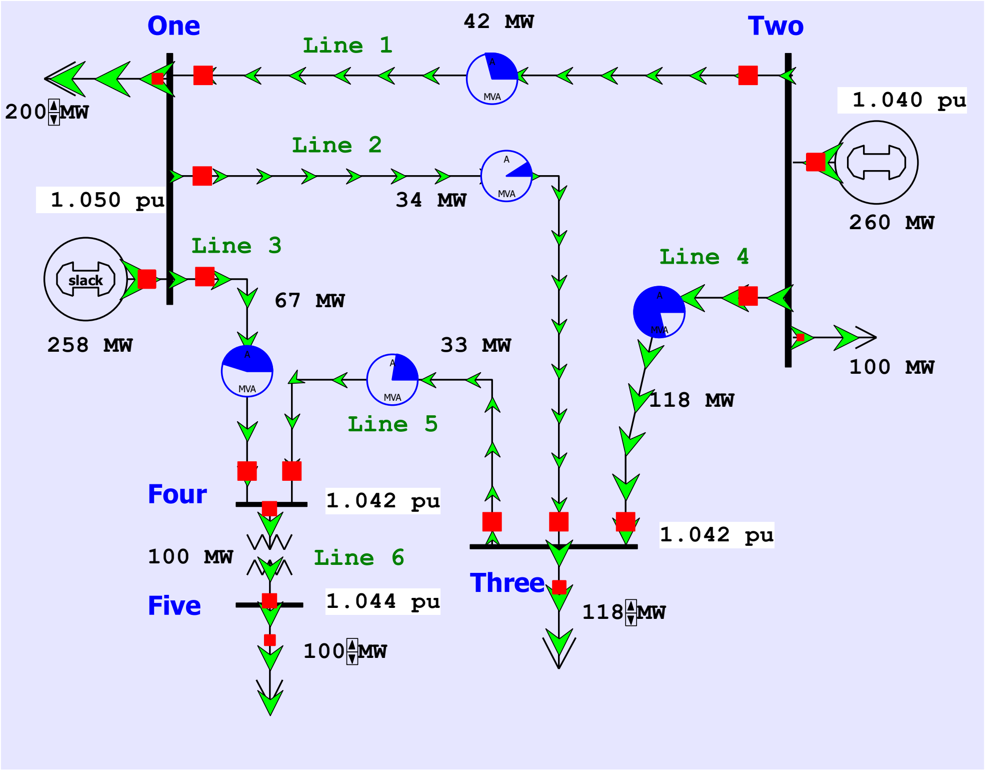
- Then as before $u_{m,n} = \min\left\{u_{m,n}^{(0)},\ u_{m,n}^{(1)}\right\}$

We would need to check all contingencies! Also, this is just a linear estimate and is not considering voltage violations.

Five Bus Example

$$w = \{2, 3, \Delta t\} \qquad \mathbf{f}^{(0)} = [42, 34, 67, 118, 33, 100]^T$$

$$\mathbf{f}^{max} = [150, 400, 150, 150, 150, 1{,}000]^T$$



Five Bus Example

Therefore, for the base case

$$u_{2,2}^{(0)} = \min_{\varphi_\ell^{(w)}>0} \left\{ \frac{f_\ell^{max} - f_\ell^{(0)}}{\varphi_\ell^{(w)}} \right\}$$

$$= \min\left\{ \frac{150-42}{0.2727}, \frac{400-34}{0.1818}, \frac{150-67}{0.0909}, \frac{150-118}{0.7273}, \frac{150-33}{0.0909} \right\}$$

$$= 44.0$$

Five Bus Example

- For the contingency case corresponding to the outage of the line 2

$$u_{2,3}^{(1)} = \min_{\left(\varphi_\ell^{(w)}\right)^2 > 0} \left\{ \frac{f_\ell^{max} - f_\ell^{(0)} - d_\ell^2 f_2^{(0)}}{\left(\varphi_\ell^{(w)}\right)^2} \right\}$$

The limiting value is line 4

$$\frac{f_\ell^{max} - f_\ell^{(0)} - d_\ell^2 f_2^{(0)}}{\left(\varphi_\ell^{(w)}\right)^2} = \frac{150 - 118 - 0.4 \times 34}{0.8}$$

Hence the UTC is limited by the contingency to 23.0

Additional Comments

- Distribution factors are defined as small signal sensitivities, but in practice, they are also used for simulating large signal cases
- Distribution factors are widely used in the operation of the electricity markets where the rapid evaluation of the impacts of each transaction on the line flows is required
- Applications to actual system show that the distribution factors provide satisfactory results in terms of accuracy
- For multiple applications that require fast turn around time, distribution factors are used very widely, particularly, in the market environment
- They do not work well with reactive power!

Least Squares

- So far we have considered the solution of $\mathbf{Ax} = \mathbf{b}$ in which $\mathbf{A}$ is a square matrix; as long as $\mathbf{A}$ is nonsingular there is a single solution
  - That is, we have the same number of equations (m) as unknowns (n)
- Many problems are overdetermined in which there more equations than unknowns ($m > n$)
  - Overdetermined systems are usually inconsistent, in which no value of $\mathbf{x}$ exactly solves all the equations
- Underdetermined systems have more unknowns than equations ($m < n$); they never have a unique solution but are usually consistent

Method of Least Squares

- The least squares method is a solution approach for determining an approximate solution for an overdetermined system
- If the system is inconsistent, then not all of the equations can be exactly satisfied
- The difference for each equation between its exact solution and the estimated solution is known as the error
- Least squares seeks to minimize the sum of the squares of the errors
- Weighted least squares allows differ weights for the equations

Least Squares Solution History

- The method of least squares developed from trying to estimate actual values from a number of measurements
- Several persons in the 1700's, starting with Roger Cotes in 1722, presented methods for trying to decrease model errors from using multiple measurements
- Legendre presented a formal description of the method in 1805; evidently Gauss claimed he did it in 1795
- Method is widely used in power systems, with state estimation the best known application, dating from Fred Schweppe's work in 1970

Least Squares and Sparsity

- In many contexts least squares is applied to problems that are not sparse. For example, using a number of measurements to optimally determine a few values
  - Regression analysis is a common example, in which a line or other curve is fit to potentially many points)
  - Each measurement impacts each model value
- In the classic power system application of state estimation the system is sparse, with measurements only directly influencing a few states
  - Power system analysis classes have tended to focus on solution methods aimed at sparse systems; we'll consider both sparse and nonsparse solution methods

Least Squares Problem

- Consider $\mathbf{Ax} = \mathbf{b}$ $\quad \mathbf{A} \in \mathbb{R}^{m\times n},\ \mathbf{x} \in \mathbb{R}^{n},\ \mathbf{b} \in \mathbb{R}^{m}$

  or

$$\begin{bmatrix} (\mathbf{a}^1)^T \\ (\mathbf{a}^2)^T \\ \vdots \\ (\mathbf{a}^m)^T \end{bmatrix} \mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{22} & \cdots & a_{2n} \\ & & \vdots & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Least Squares Solution

- We write $(\mathbf{a}^i)^T$ for the row i of **A** and $\mathbf{a}^i$ is a column vector
- Here, $m \geq n$ and the solution we are seeking is that which minimizes $\lVert \mathbf{Ax} - \mathbf{b} \rVert_p$, where *p* denotes some norm
- Since usually an overdetermined system has no exact solution, the best we can do is determine an **x** that minimizes the desired norm.

Choice of *p*

- We discuss the choice of p in terms of a specific example
- Consider the equation **Ax** = **b** with

$$\mathbf{A} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad \textbf{with } b_1 \geq b_2 \geq b_3 \geq 0$$

(hence three equations and one unknown)

- We consider three possible choices for p:

Choice of *p*

(*i*) $p = 1$

$\|Ax - b\|_1$ is minimized by $x^* = b_2$

(*ii*) $p = 2$

$\|Ax - b\|_2$ is minimized by $x^* = \dfrac{b_1 + b_2 + b_3}{3}$

(*iii*) $p = \infty$

$\|Ax - b\|_\infty$ is minimized by $x^* = \dfrac{b_1 + b_3}{2}$