# ECEN 615
# Methods of Electric Power Systems Analysis

## Lecture 15: Least Squares, State Estimation

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# Announcements

- Starting reading Chapter 9
- Homework 4 is due on Thursday October 15.

# UTC Revisited

- We can now revisit the uncommitted transfer capability (UTC) calculation using PTDFs and LODFs
- Recall trying to determine maximum transfer between two areas (or buses in our example)
- For base case maximums are quickly determined with PTDFs

$$u_{m,n}^{(0)} = \min_{\varphi_\ell^{(w)}>0} \left\{ \frac{f_\ell^{max} - f_\ell^{(0)}}{\varphi_\ell^{(w)}} \right\}$$

Note we are ignoring zero (or small) PTDFs; would also need to consider flow reversal

# UTC Revisited

- For the contingencies we use

$$u_{m,n}^{(1)} = \min_{\left(\varphi_\ell^{(w)}\right)^k > 0} \left\{ \frac{f_\ell^{max} - f_\ell^{(0)} - d_\ell^k f_k^{(0)}}{\left(\varphi_\ell^{(w)}\right)^k} \right\}$$

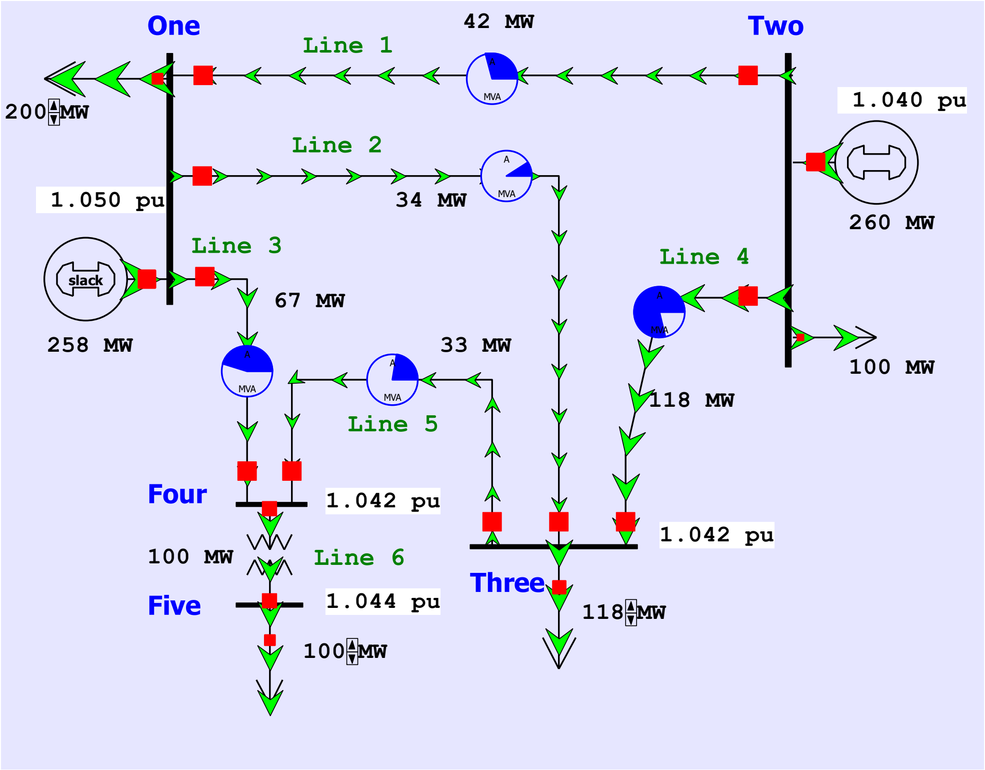
- Then as before $u_{m,n} = \min\left\{u_{m,n}^{(0)},\ u_{m,n}^{(1)}\right\}$

We would need to check all contingencies! Also, this is just a linear estimate and is not considering voltage violations.

# Five Bus Example

$$w = \{2, 3, \Delta t\} \qquad \mathbf{f}^{(0)} = [42, 34, 67, 118, 33, 100]^T$$

$$\mathbf{f}^{max} = [150, 400, 150, 150, 150, 1{,}000]^T$$



# Five Bus Example

Therefore, for the base case

$$u_{2,2}^{(0)} = \min_{\varphi_\ell^{(w)}>0} \left\{ \frac{f_\ell^{max} - f_\ell^{(0)}}{\varphi_\ell^{(w)}} \right\}$$

$$= \min \left\{ \frac{150-42}{0.2727}, \frac{400-34}{0.1818}, \frac{150-67}{0.0909}, \frac{150-118}{0.7273}, \frac{150-33}{0.0909} \right\}$$

$$= 44.0$$

# Five Bus Example

- For the contingency case corresponding to the outage of the line 2

$$u_{2,3}^{(1)} = \min_{\left(\varphi_\ell^{(w)}\right)^2 > 0} \left\{ \frac{f_\ell^{max} - f_\ell^{(0)} - d_\ell^{2} f_2^{(0)}}{\left(\varphi_\ell^{(w)}\right)^2} \right\}$$

The limiting value is line 4

$$\frac{f_\ell^{max} - f_\ell^{(0)} - d_\ell^{2} f_2^{(0)}}{\left(\varphi_\ell^{(w)}\right)^2} = \frac{150 - 118 - 0.4 \times 34}{0.8}$$

Hence the UTC is limited by the contingency to 23.0

# Additional Comments

- Distribution factors are defined as small signal sensitivities, but in practice, they are also used for simulating large signal cases
- Distribution factors are widely used in the operation of the electricity markets where the rapid evaluation of the impacts of each transaction on the line flows is required
- Applications to actual system show that the distribution factors provide satisfactory results in terms of accuracy
- For multiple applications that require fast turn around time, distribution factors are used very widely, particularly, in the market environment
- They do not work well with reactive power!

# Least Squares

- So far we have considered the solution of **Ax** = **b** in which **A** is a square matrix; as long as **A** is nonsingular there is a single solution
  - That is, we have the same number of equations (m) as unknowns (n)
- Many problems are overdetermined in which there more equations than unknowns (m > n)
  - Overdetermined systems are usually inconsistent, in which no value of **x** exactly solves all the equations
- Underdetermined systems have more unknowns than equations (m < n); they never have a unique solution but are usually consistent

# Method of Least Squares

- The least squares method is a solution approach for determining an approximate solution for an overdetermined system
- If the system is inconsistent, then not all of the equations can be exactly satisfied
- The difference for each equation between its exact solution and the estimated solution is known as the error
- Least squares seeks to minimize the sum of the squares of the errors
- Weighted least squares allows differ weights for the equations

# Least Squares Solution History

- The method of least squares developed from trying to estimate actual values from a number of measurements
- Several persons in the 1700's, starting with Roger Cotes in 1722, presented methods for trying to decrease model errors from using multiple measurements
- Legendre presented a formal description of the method in 1805; evidently Gauss claimed he did it in 1795
- Method is widely used in power systems, with state estimation the best known application, dating from Fred Schweppe's work in 1970

# Least Squares and Sparsity

- In many contexts least squares is applied to problems that are not sparse. For example, using a number of measurements to optimally determine a few values
  - Regression analysis is a common example, in which a line or other curve is fit to potentially many points)
  - Each measurement impacts each model value
- In the classic power system application of state estimation the system is sparse, with measurements only directly influencing a few states
  - Power system analysis classes have tended to focus on solution methods aimed at sparse systems; we'll consider both sparse and nonsparse solution methods

# Least Squares Problem

- Consider $\mathbf{A}\mathbf{x} = \mathbf{b}$ $\qquad \mathbf{A} \in \mathbb{R}^{m\times n},\ \mathbf{x} \in \mathbb{R}^{n},\ \mathbf{b} \in \mathbb{R}^{m}$

  or

$$\begin{bmatrix} (\mathbf{a}^1)^T \\ (\mathbf{a}^2)^T \\ \vdots \\ (\mathbf{a}^m)^T \end{bmatrix} \mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{22} & \dots & a_{2n} \\ & & \vdots & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

# Least Squares Solution

- We write $(\mathbf{a}^i)^T$ for the row i of $\mathbf{A}$ and $\mathbf{a}^i$ is a column vector
- Here, $m \geq n$ and the solution we are seeking is that which minimizes $\|\mathbf{Ax} - \mathbf{b}\|_p$, where $p$ denotes some norm
- Since usually an overdetermined system has no exact solution, the best we can do is determine an $\mathbf{x}$ that minimizes the desired norm.

# Choice of *p*

- We discuss the choice of $p$ in terms of a specific example
- Consider the equation $\mathbf{Ax} = \mathbf{b}$ with

$$\mathbf{A} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad \textbf{with } b_1 \geq b_2 \geq b_3 \geq 0$$

(hence three equations and one unknown)

- We consider three possible choices for $p$:

# Choice of *p*

$(i)\quad p = 1$

$\|Ax - b\|_1$ is minimized by $x^* = b_2$

$(ii)\quad p = 2$

$\|Ax - b\|_2$ is minimized by $x^* = \dfrac{b_1 + b_2 + b_3}{3}$

$(iii)\quad p = \infty$

$\|Ax - b\|_\infty$ is minimized by $x^* = \dfrac{b_1 + b_3}{2}$

# The Least Squares Problem

- In general, $\| \mathbf{Ax} - \mathbf{b} \|_p$ is not differentiable for $p = 1$ or $p = \infty$
- The choice of $p = 2$ (Euclidean norm) has become well established given its least-squares fit interpretation
- The problem $\min_{\mathbf{x} \in ¡^n} \| \mathbf{Ax} - \mathbf{b} \|_2$ is tractable for 2 major reasons
  - First, the function is differentiable

$$\phi(\mathbf{x}) = \frac{1}{2} \| \mathbf{Ax} - \mathbf{b} \|_2^2 = \frac{1}{2} \sum_{i=1}^{m} \left[ \left( \mathbf{a}^i \right)^T \mathbf{x} - b_i \right]^2$$

# The Least Squares Problem, cont.

– Second, the Euclidean norm is preserved under orthogonal transformations:

$$\left\| \left(\mathbf{Q}^T\mathbf{A}\right)\mathbf{x} - \mathbf{Q}^T\mathbf{b} \right\|_2 = \left\| \mathbf{A}\mathbf{x} - \mathbf{b} \right\|_2$$

with **Q** an arbitrary orthogonal matrix; that is, **Q** satisfies

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I} \qquad \mathbf{Q} \in \mathbb{R}^{n \times n}$$

# The Least Squares Problem, cont.

- We introduce next the basic underlying assumption: **A** is full rank, i.e., the columns of **A** constitute a set of linearly independent vectors
- This assumption implies that the rank of **A** is n because n ≤ m since we are dealing with an overdetermined system
- Fact: The least squares solution $\mathbf{x}^*$ satisfies

$$\mathbf{A}^T \mathbf{A} \mathbf{x}^* = \mathbf{A}^T \mathbf{b}$$

# Proof of Fact

- Since by definition the least squares solution $\mathbf{x}^*$ minimizes $\phi(\bullet)$ at the optimum, the derivative of this function zero:

$$\phi(\mathbf{x}) = \frac{1}{2}\|\mathbf{Ax}-\mathbf{b}\|_2^2 = \frac{1}{2}\left(\mathbf{x}^T\mathbf{A}^T\mathbf{Ax}-\mathbf{x}^T\mathbf{A}^T\mathbf{b}-\mathbf{b}^T\mathbf{A}\,\mathbf{x}+\mathbf{b}^T\mathbf{b}\right)$$

$$\mathbf{0} = \left.\frac{\partial\,\phi(\mathbf{x})}{\partial\mathbf{x}}\right|_{\mathbf{x}^*} = \left.\frac{\partial}{\partial\mathbf{x}}\left\{\frac{1}{2}\left(\mathbf{x}^T\mathbf{A}^T\mathbf{Ax}-\mathbf{x}^T\mathbf{A}^T\mathbf{b}-\mathbf{b}^T\mathbf{A}\,\mathbf{x}+\mathbf{b}^T\mathbf{b}\right)\right\}\right|_{\mathbf{x}^*}$$

$$= \left.\frac{\partial}{\partial\mathbf{x}}\left\{\frac{1}{2}\left(\mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} - 2\mathbf{x}^T\mathbf{A}^T\mathbf{b} + \mathbf{b}^T\mathbf{b}\right)\right\}\right|_{\mathbf{x}^*}$$

$$= \mathbf{A}^T\mathbf{A}\mathbf{x}^* - \mathbf{A}^T\mathbf{b}$$

# Implications

- This underlying assumption implies that

  $\mathbf{A}$ is full rank $\Leftrightarrow \exists\ \mathbf{x} \neq \mathbf{0} \ \ni\ \mathbf{Ax} \neq \mathbf{0}$

- Therefore, the fact that $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is positive definite (*p.d.*) follows from considering any $\mathbf{x} \neq \mathbf{0}$ and evaluating

  $$\mathbf{x}^{T}\mathbf{A}^{T}\mathbf{A}\,\mathbf{x} = \|\mathbf{A}\mathbf{x}\|_2^2 > 0,$$

  which is the definition of a *p.d.* matrix

- We use the shorthand $\mathbf{A}^{\mathrm{T}}\mathbf{A} > \mathbf{0}$ for $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ being a symmetric, positive definite matrix

# Implications

- The underlying assumption that **A** is full rank and therefore $\mathbf{A}^T\mathbf{A}$ is *p.d.* implies that there exists a unique least squares solution
- Note: we use the inverse in a conceptual, rather than a computational, sense

$$\mathbf{x}^* = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}$$

- The below formulation is known as the normal equations, with the solution conceptually straightforward

$$\left(\mathbf{A}^T\mathbf{A}\right)\mathbf{x} = \mathbf{A}^T\mathbf{b}$$

# Example: Curve Fitting

- Say we wish to fit five points to a polynomial curve of the form

$$\mathrm{f}(t, \mathbf{x}) = x_1 + x_2 t + x_3 t^2$$

- This can be written as

$$\mathbf{A}\mathbf{x} = \mathbf{y} = \begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \\ 1 & t_4 & t_4^2 \\ 1 & t_5 & t_5^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

# Example: Curve Fitting

- Say the points are **t** =[0,1,2,3,4] and **y** = [0,2,4,5,4]. Then

$$\mathbf{Ax} = \mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 5 \\ 4 \end{bmatrix}$$

$$\mathbf{x}^* = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b} = \begin{bmatrix} 0.886 & 0.257 & -0.086 & -0.143 & 0.086 \\ -0.771 & 0.186 & 0.571 & 0.386 & -0.371 \\ 0.143 & -0.071 & -0.143 & -0.071 & 0.143 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 4 \\ 5 \\ 4 \end{bmatrix}$$

$$\mathbf{x}^* = \begin{bmatrix} -0.2 \\ 3.1 \\ -0.5 \end{bmatrix}$$

# Implications

- An important implication of positive definiteness is that we can factor $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ since $\mathbf{A}^{\mathrm{T}}\mathbf{A} > \mathbf{0}$

$$\mathbf{A}^{T}\mathbf{A} = \mathbf{U}^{T}\mathbf{D}\,\mathbf{U} = \mathbf{U}^{T}\mathbf{D}^{1/2}\mathbf{D}^{1/2}\mathbf{U} = \mathbf{G}^{T}\mathbf{G}$$

- The expression $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{G}^{\mathrm{T}}\mathbf{G}$ is called the Cholesky factorization of the symmetric positive definite matrix $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

# A Least Squares Solution Algorithm

*Step* 1: Compute the lower triangular part of $\mathbf{A}^T\mathbf{A}$

*Step* 2: Obtain the Cholesky Factorization $\mathbf{A}^T\mathbf{A} = \mathbf{G}^T\mathbf{G}$

*Step* 3: Compute $\mathbf{A}^T\mathbf{b} = \hat{\mathbf{b}}$

*Step* 4: Solve for $\mathbf{y}$ using forward substitution in

$$\mathbf{G}^T\mathbf{y} = \hat{\mathbf{b}}$$

and for $\mathbf{x}$ using backward substitution in

$$\mathbf{G}\,\mathbf{x} = \mathbf{y}$$

Note, our standard LU factorization approach would work; we can just solve it twice as fast by taking advantage of it being a symmetric matrix

# Practical Considerations

- The two key problems that arise in practice with the triangularization procedure are:
  – First, while **A** maybe sparse, $\mathbf{A}^T\mathbf{A}$ is much less sparse and consequently requires more computing resources for the solution
    - In particular, with $\mathbf{A}^T\mathbf{A}$ **s**econd neighbors are now connected! Large networks are still sparse, just not as sparse
  – Second, $\mathbf{A}^T\mathbf{A}$ may actually be numerically less well-conditioned than **A**

# Loss of Sparsity Example

- Assume the **B** matrix for a network is

$$\mathbf{B} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

- Then $\mathbf{B}^T\mathbf{B}$ is $\mathbf{B}^T\mathbf{B} = \begin{bmatrix} 2 & -3 & 1 & 0 \\ -3 & 6 & -4 & 1 \\ 1 & -4 & 6 & -3 \\ 0 & 1 & -3 & 2 \end{bmatrix}$

- Second neighbors are now connected!

# Numerical Conditioning

- To understand the point on numerical ill-conditioning, we need to introduce terminology
- We define the norm of a matrix $\mathbf{B} \in \mathbb{R}^{m \times n}$ to be

$$\|\mathbf{B}\| = \max_{\mathbf{x} \neq \mathbf{0}} \left\{ \frac{\|\mathbf{B}\mathbf{x}\|}{\|\mathbf{x}\|} \right\}$$

$$= \textit{maximum stretching of the matrix } \mathbf{B}$$

- This is the maximum singular value of **B**

# Numerical Conditioning Example

- Say we have the matrix

$$\mathbf{B} = \begin{bmatrix} 10 & 0 \\ 0 & 0.1 \end{bmatrix}$$

- What value of **x** with a norm of 1 that maximizes $\|\mathbf{Bx}\|$?
- What value of **x** with a norm of 1 that minimizes $\|\mathbf{Bx}\|$?

$$\|\mathbf{B}\| = \max_{\mathbf{x} \neq \mathbf{0}} \left\{ \frac{\|\mathbf{B}\,\mathbf{x}\|}{\|\mathbf{x}\|} \right\}$$

$$= \textit{maximum stretching of the matrix } \mathbf{B}$$

# Numerical Conditioning

$$= \max_i \left\{ \sqrt{\lambda_i} \,,\ \lambda_i \text{ is an eigenvalue of } \underline{B}^T \underline{B} \right\},$$

i.e., $\lambda_i$ is a root of the polynomial

$$p(\lambda) = det\left[ \mathbf{B}^T \mathbf{B} - \lambda \mathbf{I} \right]$$

Keep in mind the eigenvalues of a p.d. matrix are positive

- In other words, the $\ell_2$ norm of **B** is the square root of the largest eigenvalue of $\mathbf{B}^T\mathbf{B}$

# Numerical Conditioning

- The conditioning number of a matrix **B** is defined as

$$\kappa(\mathbf{B}) = \|\mathbf{B}\| \, \|\mathbf{B}^{-1}\| = \frac{|\sigma_{max}(\mathbf{B})|}{|\sigma_{min}(\mathbf{B})|} \Bigg\} \quad \textit{the max / min stretching ratio of the matrix } \mathbf{B}$$

- A well–conditioned matrix has a small value of $\kappa(\mathbf{B})$, close to 1; the larger the value of $\kappa(\mathbf{B})$, the more pronounced is the ill-conditioning

# Power System State Estimation (SE)

- The need is because in power system operations there is a desire to do "what if" studies based upon the actual "state" of the electric grid
  - An example is an online power flow or contingency analysis
- Overall goal of SE is to come up with a power flow model for the present "state" of the power system based on the actual system measurements
- SE assumes the topology and parameters of the transmission network are mostly known
- Measurements from SCADA and increasingly PMUs
- Overview is given in ECEN 615; more details in 614

# Power System State Estimation

- Problem can be formulated in a nonlinear, weighted least squares form as

$$\min J(\mathbf{x}) = \sum_{i=1}^{m} \frac{\left[ z_i - f_i(\mathbf{x}) \right]^2}{\sigma_i^2}$$

where J($\mathbf{x}$) is the scalar cost function, **x** are the state variables (primarily bus voltage magnitudes and angles), $z_i$ are the m measurements, **f**($\mathbf{x}$) relates the states to the measurements and $\sigma_i$ is the assumed standard deviation for each measurement

# Assumed Error

- Hence the goal is to decrease the error between the measurements and the assumed model states **x**
- The $\sigma_i$ term weighs the various measurements, recognizing that they can have vastly different assumed errors

$$\min J(\mathbf{x}) = \sum_{i=1}^{m} \frac{\left[z_i - f_i(\mathbf{x})\right]^2}{\sigma_i^2}$$

- Measurement error is assumed Gaussian (whether it is or not is another question); outliers (bad measurements) are often removed

# State Estimation for Linear Functions

- First we'll consider the linear problem. That is where

$$\mathbf{z}^{meas} - \mathbf{f}(\mathbf{x}) = \mathbf{z}^{meas} - \mathbf{H}\mathbf{x}$$

- Let **R** be defined as the diagonal matrix of the variances (square of the standard deviations) for each of the measurements

$$\mathbf{R} = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_m^2 \end{bmatrix}$$

# State Estimation for Linear Functions

- We then differentiate J(**x**) w.r.t. **x** to determine the value of **x** that minimizes this function

$$J(\mathbf{x}) = \left[\mathbf{z}^{meas} - \mathbf{H}\mathbf{x}\right]^T \mathbf{R}^{-1} \left[\mathbf{z}^{meas} - \mathbf{H}\mathbf{x}\right]$$

$$\nabla J(\mathbf{x}) = -2\mathbf{H}^T\mathbf{R}^{-1}\mathbf{z}^{meas} + 2\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{x}$$

At the minimum we have $\nabla J(\mathbf{x}) = \mathbf{0}$. So solving for $\mathbf{x}$ gives

$$\mathbf{x} = \left[\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right]^{-1} \mathbf{H}^T\mathbf{R}^{-1}\mathbf{z}^{meas}$$

# Simple DC System Example

- Say we have a two bus power system that we are solving using the dc approximation. Say the line's per unit reactance is j0.1. Say we have power measurements at both ends of the line. For simplicity assume **R**=**I**. We would then like to estimate the bus angles. Then

$$z_1 = P_{12} = \frac{\theta_1 - \theta_2}{0.1} = 2.2, \quad z_2 = -2.0 = P_{21} = \frac{\theta_2 - \theta_1}{0.1}$$

$$\mathbf{x} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \mathbf{H} = \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix}, \mathbf{H}^T\mathbf{H} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix}$$

We have a problem since $\mathbf{H}^T\mathbf{H}$ is singular. This is because of lack of an angle reference.

# Simple DC System Example, cont.

- Say we directly measure $\theta_1$ (with a PMU) to be zero; set this as the third measurement. Then

$$z_1 = P_{12} = \frac{\theta_1 - \theta_2}{0.1} = 2.2, \quad z_2 = -2.0 = P_{21} = \frac{\theta_2 - \theta_1}{0.1}, \quad z_3 = 0$$

$$\mathbf{x} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} 2.2 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 10 & -10 \\ -10 & 10 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{H}^T\mathbf{H} = \begin{bmatrix} 201 & -200 \\ -200 & 200 \end{bmatrix}$$

$$\mathbf{x} = \left[\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right]^{-1}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{z}^{meas}$$

$$\mathbf{x} = \begin{bmatrix} 201 & -200 \\ -200 & 200 \end{bmatrix}^{-1} \begin{bmatrix} 10 & -10 & 1 \\ -10 & 10 & 0 \end{bmatrix} \begin{bmatrix} 2.2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.21 \end{bmatrix}$$

Note that the angles are in radians

# Nonlinear Formulation

- A regular ac power system is nonlinear, so we need to use an iterative solution approach. This is similar to the Newton power flow. Here assume m measurements and n state variables (usually bus voltage magnitudes and angles) Then the Jacobian is the **H** matrix

$$\mathbf{H}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{x_1} & \cdots & \frac{\partial f_1}{x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{x_1} & \cdots & \frac{\partial f_m}{x_n} \end{bmatrix}$$

# Measurement Example

- Assume we measure the real and reactive power flowing into one end of a transmission line; then the $z_i$-$f_i(\mathbf{x})$ functions for these two are

$$P_{ij}^{meas} - \left[ -V_i^2 G_{ij} + V_i V_j \left( G_{ij} \cos\left(\theta_i - \theta_j\right) + B_{ij} \sin\left(\theta_i - \theta_j\right)\right)\right]$$

$$Q_{ij}^{meas} - \left[ V_i^2 \left( B_{ij} + \frac{B_{cap}}{2} \right) + V_i V_j \left( G_{ij} \sin\left(\theta_i - \theta_j\right) - B_{ij} \cos\left(\theta_i - \theta_j\right)\right)\right]$$

  – Two measurements for four unknowns

- Other measurements, such as the flow at the other end, and voltage magnitudes, add redundancy

# SE Iterative Solution Algorithm

- We then make an initial guess of **x**, $\mathbf{x}^{(0)}$ and iterate, calculating $\Delta\mathbf{x}$ each iteration

$$\Delta\mathbf{x} = \left[\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right]^{-1}\mathbf{H}^T\mathbf{R}^{-1}\begin{bmatrix} z_1 - f_1(\mathbf{x}) \\ \vdots \\ z_m - f_m(\mathbf{x}) \end{bmatrix}$$

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \Delta\mathbf{x}$$

This is exactly the least squares form developed earlier with $\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}$ an n by n matrix. This could be solved with Gaussian elimination, but this isn't preferred because the problem is often ill-conditioned

Keep in mind that H is no longer constant, but varies as x changes. often ill-conditioned

# Nonlinear SE Solution Algorithm, Book Figure 9.11



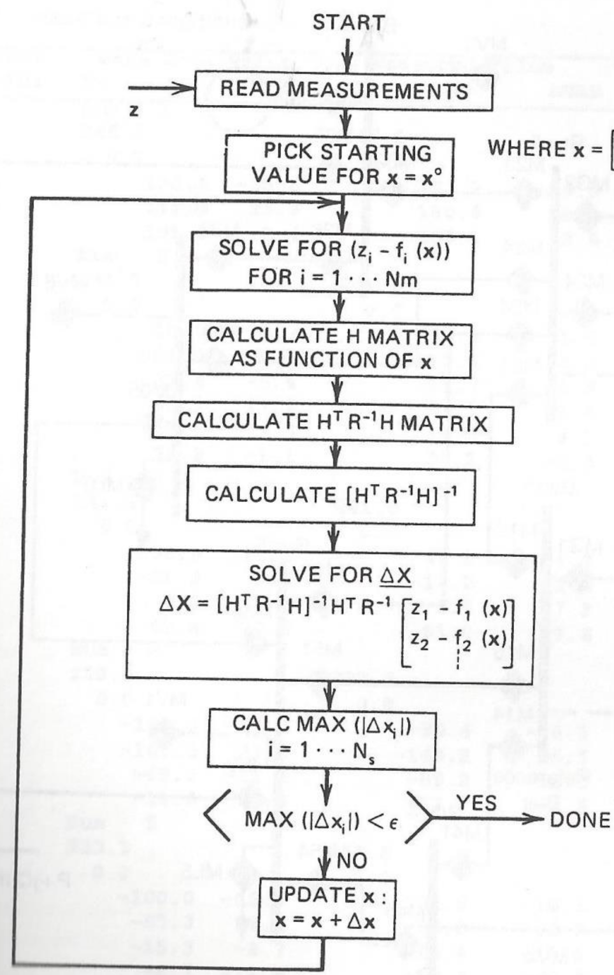



**FIGURE 9.11** State estimation solution algorithm.

# Example: Two Bus Case

- Assume a two bus case with a generator supplying a load through a single line with x=0.1 pu. Assume measurements of the p/q flow on both ends of the line (into line positive), and the voltage magnitude at both the generator and the load end. So $B_{12}$ = $B_{21}$=10.0

$$P_{ij}^{meas} - \left[ V_i V_j \left( B_{ij} \sin\left( \theta_i - \theta_j \right) \right) \right]$$

$$Q_{ij}^{meas} - \left[ V_i^2 B_{ij} + V_i V_j \left( -B_{ij} \cos\left( \theta_i - \theta_j \right) \right) \right]$$

$$V_i^{meas} - V_i = 0$$

We need to assume a reference angle unless we directly measuring phase

# Example: Two Bus Case

- Let $$\mathbf{z}^{meas} = \begin{bmatrix} P_{12} \\ Q_{12} \\ P_{21} \\ Q_{21} \\ V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 2.02 \\ 1.5 \\ -1.98 \\ -1 \\ 1.01 \\ 0.87 \end{bmatrix} \quad x^0 = \begin{bmatrix} V_1 \\ \theta_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \sigma_i = 0.01$$

We assume an angle reference of $\theta_1=0$

$$H(\mathbf{x}) = \begin{bmatrix} V_2 10\sin(-\theta_2) & -V_1V_2 10\cos(-\theta_2) & V_1 10\sin(-\theta_2) \\ 20V_1 - V_2 10\cos(-\theta_2) & -V_1V_2 10\sin(-\theta_2) & -V_1 10\cos(-\theta_2) \\ V_2 10\sin(\theta_2) & V_1V_2 10\cos(\theta_2) & V_1 10\sin(\theta_2) \\ -V_2 10\cos(\theta_2) & V_1V_2 10\sin(\theta_2) & 20V_2 - V_1 10\cos(\theta_2) \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Example: Two Bus Case

- With a flat start guess we get

$$H(\mathbf{x}^0) = \begin{bmatrix} 0 & -10 & 0 \\ 10 & 0 & -10 \\ 0 & 10 & 0 \\ -10 & 0 & 10 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{z} - \mathbf{f}(\mathbf{x}^0) = \begin{bmatrix} 2.02 \\ 1.5 \\ -1.98 \\ -1 \\ 0.01 \\ -0.13 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} 0.0001 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0001 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0001 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0001 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0001 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0001 \end{bmatrix}$$

# Example: Two Bus Case

$$\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H} = 1e^6 \times \begin{bmatrix} 2.01 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 2.01 \end{bmatrix}$$

$$\mathbf{x}^1 = \mathbf{x}^0 + \left[\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right]^{-1}\mathbf{H}^T\mathbf{R}^{-1}\begin{bmatrix} 2.02 \\ 1.5 \\ -1.98 \\ -1 \\ 0.01 \\ -0.13 \end{bmatrix} = \begin{bmatrix} 1.003 \\ -0.2 \\ 0.8775 \end{bmatrix}$$

# Assumed SE Measurement Accuracy

- The assumed measurement standard deviations can have a significant impact on the resultant solution, or even whether the SE converges
- The assumption is a Gaussian (normal) distribution of the error with no bias

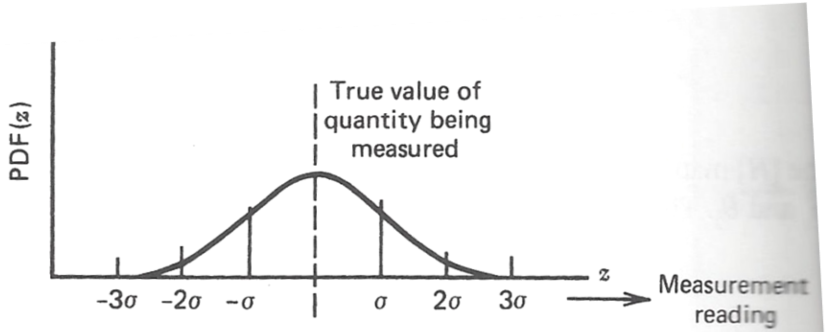



FIGURE 9.8 Normal distribution of meter errors.

# SE Observability

- In order to estimate all n states we need at least n measurements. However, where the measurements are located is also important, a topic known as observability
  - In order for a power system to be fully observable usually we need to have a measurement available no more than one bus away
  - At buses we need to have at least measurements on all the injections into the bus except one (including loads and gens)
  - Loads are usually flows on feeders, or the flow into a transmission to distribution transformer
  - Generators are usually just injections from the GSU

# Pseudo Measurements

- Pseudo measurements are used at buses in which there is no load or generation; that is, the net injection into the bus is know with high accuracy to be zero
  - In order to enforce the net power balance at a bus we need to include an explicit net injection measurement
- To increase observability sometimes estimated values are used for loads, shunts and generator outputs
  - These “measurements” are represented as having a higher much standard deviation

# SE Observability Example