ECEN 667
Power System Stability

Lecture 21: Modal Analysis

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Announcements

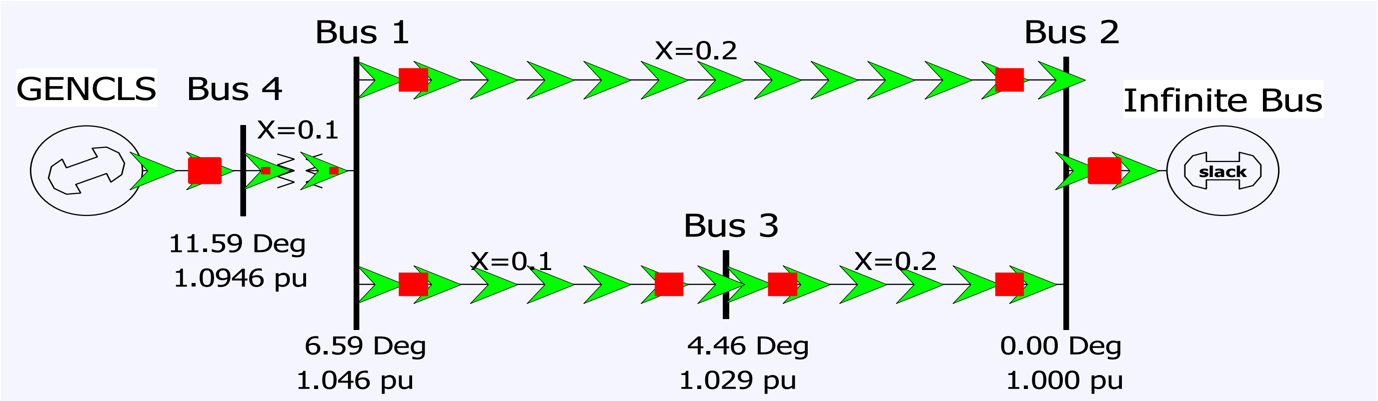
- Read Chapter 8
- Homework 7 is posted; due on Tuesday Nov 28
- Final is as per TAMU schedule. That is, Friday Dec 8 from 3 to 5pm

Single Machine Infinite Bus

- A quite useful analysis technique is to consider the small signal stability associated with a single generator connected to the rest of the system through an equivalent transmission line
- Driving point impedance looking into the system is used to calculate the equivalent line's impedance
  - The Z_{ii} value can be calculated quite quickly using sparse vector methods
- Rest of the system is assumed to be an infinite bus with its voltage set to match the generator's real and reactive power injection and voltage

Small SMIB Example

- As a small example, consider the 4 bus system shown below, in which bus 2 really is an infinite bus



- To get the SMIB for bus 4, first calculate Z_{44}

$$Y_{bus} = j\begin{bmatrix} -25 & 0 & 10 & 10 \\ 0 & 1 & 0 & 0 \\ 10 & 0 & -15 & 0 \\ 10 & 0 & 0 & -13.33 \end{bmatrix} \rightarrow Z_{44} = j0.1269$$

Z_{44} is Z_{th} in parallel with $jX'_{d,4}$ (which is j0.3) so Z_{th} is j0.22

Example: Bus 4 with GENROU Model

- The eigenvalues can be calculated for any set of generator models
- This example replaces the bus 4 generator classical machine with a GENROU model
  - There are now six eigenvalues, with the dominate response coming from the electro-mechanical mode with a frequency of 1.83 Hz, and damping of 6.92%

Generator Information (on Generator MVA Base)

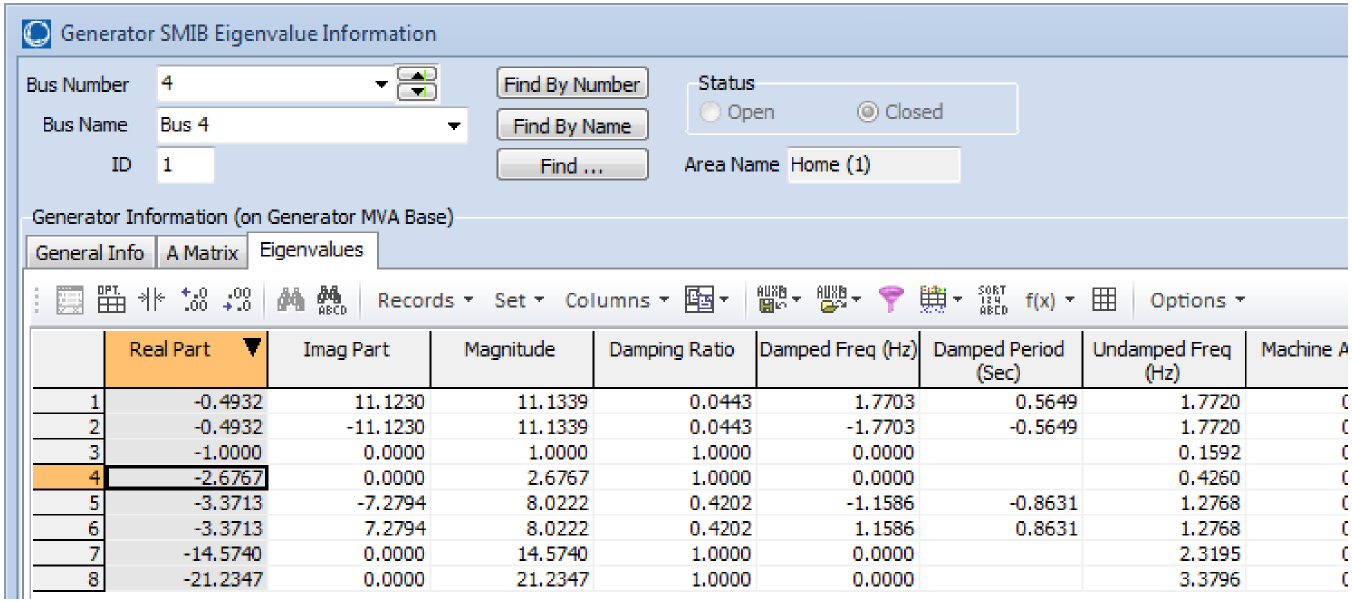
General Info | A Matrix | Eigenvalues

Records Set Columns f(x) Options

	Real Part	Imag Part	Magnitude	Damping Ratio	Damped Freq (Hz)	Damped Period (Sec)	Undamped Freq (Hz)	Machine Angle	Machi
1	-21.2494	0.0000	21.2494	1.0000	0.0000		3.3819	0.0159	
2	-0.8016	11.5521	11.5798	0.0692	1.8386	0.5439	1.8430	0.7055	
3	-0.8016	-11.5521	11.5798	0.0692	-1.8386	-0.5439	1.8430	0.7055	
4	-14.2252	0.0000	14.2252	1.0000	0.0000		2.2640	0.0044	
5	-3.7124	0.0000	3.7124	1.0000	0.0000		0.5909	0.0154	
6	-0.4231	0.0000	0.4231	1.0000	0.0000		0.0673	0.0027	

Example: Bus 4 with GENROU Model and Exciter

- Adding an relatively slow EXST1 exciter adds additional states (with K_A=200, T_A=0.2)
  - As the initial reactive power output of the generator is decreased, the system becomes unstable



	Real Part	Imag Part	Magnitude	Damping Ratio	Damped Freq (Hz)	Damped Period (Sec)	Undamped Freq (Hz)	Machine A
1	-0.4932	11.1230	11.1339	0.0443	1.7703	0.5649	1.7720	
2	-0.4932	-11.1230	11.1339	0.0443	-1.7703	-0.5649	1.7720	
3	-1.0000	0.0000	1.0000	1.0000	0.0000		0.1592	
4	-2.6767	0.0000	2.6767	1.0000	0.0000		0.4260	
5	-3.3713	-7.2794	8.0222	0.4202	-1.1586	-0.8631	1.2768	
6	-3.3713	7.2794	8.0222	0.4202	1.1586	0.8631	1.2768	
7	-14.5740	0.0000	14.5740	1.0000	0.0000		2.3195	
8	-21.2347	0.0000	21.2347	1.0000	0.0000		3.3796	

Case is saved as B4_GENROU_Sat_SMIB

Example: Bus 4 with GENROU Model and Exciter

- With Q_4 = 25 Mvar the eigenvalues are

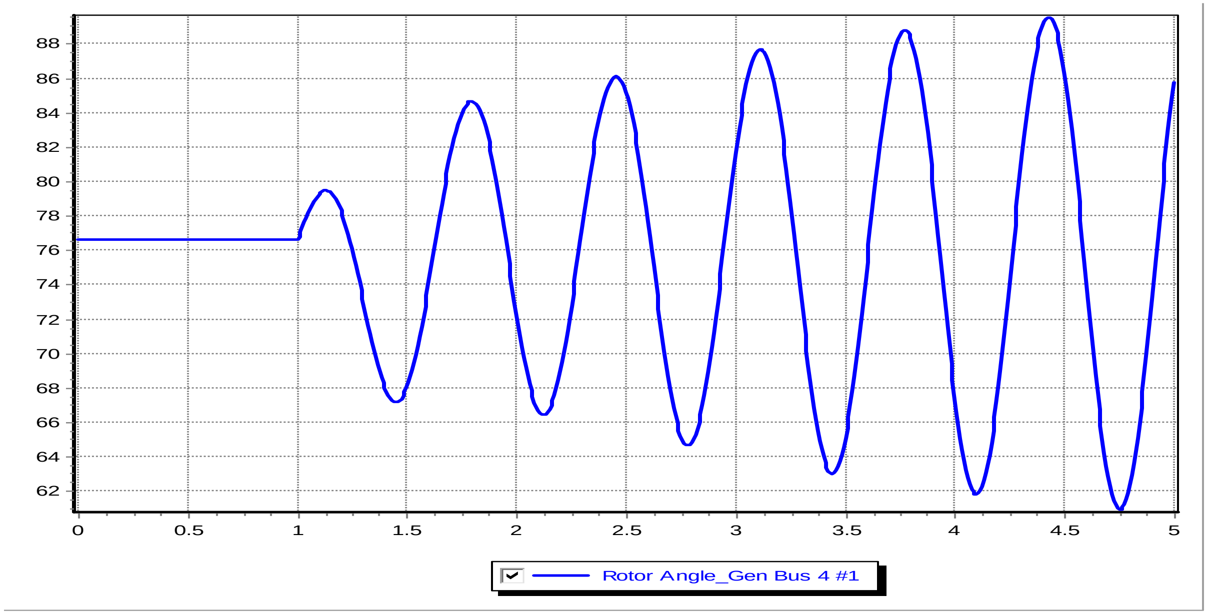
	Real Part	Imag Part	Magnitude	Damping Ratio	Damped Freq (Hz)	Damped Period (Sec)	Undamped Freq (Hz)	Machine A
1	-0.1239	-10.3955	10.3962	0.0119	-1.6545	-0.6044	1.6546	(
2	-0.1239	10.3955	10.3962	0.0119	1.6545	0.6044	1.6546	(
3	-1.0000	0.0000	1.0000	1.0000	0.0000		0.1592	(
4	-2.6586	0.0000	2.6586	1.0000	0.0000		0.4231	(
5	-3.5938	-6.8580	7.7426	0.4642	-1.0915	-0.9162	1.2323	(
6	-3.5938	6.8580	7.7426	0.4642	1.0915	0.9162	1.2323	(
7	-14.5078	0.0000	14.5078	1.0000	0.0000		2.3090	(
8	-21.4739	0.0000	21.4739	1.0000	0.0000		3.4177	(

- And with Q_4=0 Mvar the eigenvalues are

	Real Part	Imag Part	Magnitude	Damping Ratio	Damped Freq (Hz)	Damped Period (Sec)	Undamped Freq (Hz)	Machine A
1	0.2704	-9.5336	9.5374	-0.0283	-1.5173	-0.6591	1.5179	(
2	0.2704	9.5336	9.5374	-0.0283	1.5173	0.6591	1.5179	(
3	-1.0000	0.0000	1.0000	1.0000	0.0000		0.1592	(
4	-3.0137	0.0000	3.0137	1.0000	0.0000		0.4796	(
5	-3.6849	-6.4281	7.4094	0.4973	-1.0231	-0.9775	1.1792	(
6	-3.6849	6.4281	7.4094	0.4973	1.0231	0.9775	1.1792	(
7	-14.4234	0.0000	14.4234	1.0000	0.0000		2.2956	(
8	-21.6978	0.0000	21.6978	1.0000	0.0000		3.4533	(

Example: Bus 4 with GENROU Model and Exciter

- Graph shows response following a short fault when Q4 is 0 Mvar



- This motivates trying to get additional insight into how to increase system damping, which is the goal of modal analysis

Modal Analysis - Comments

- Modal analysis (analysis of small signal stability through eigenvalue analysis) is at the core of SSA software
- In Modal Analysis one looks at:
  - Eigenvalues
  - Eigenvectors (left or right)
  - Participation factors
  - Mode shape

Goal is to determine how the various parameters affect the response of the system

- Power System Stabilizer (PSS) design in a multi-machine context is done using modal analysis method.

Eigenvalues, Right Eigenvectors

- For an n by n matrix **A** the eigenvalues of **A** are the roots of the characteristic equation:

$$\det[\mathbf{A}-\lambda\mathbf{I}] = |\mathbf{A}-\lambda\mathbf{I}| = 0$$

- Assume $\lambda_1 \ldots \lambda_n$ as distinct (no repeated eigenvalues).
- For each eigenvalue λ_i there exists an eigenvector such that:

$$\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

- $\mathbf{v}_i$ is called a right eigenvector
- If λ_i is complex, then $\mathbf{v}_i$ has complex entries

Left Eigenvectors

- For each eigenvalue λ_i there exists a left eigenvector $\mathbf{w}_i$ such that:

$$\mathbf{w}_i^t \mathbf{A} = \mathbf{w}_i^t \lambda_i$$

- Equivalently, the left eigenvector is the right eigenvector of $\mathbf{A}^T$; that is,

$$\mathbf{A}^t \mathbf{w}_i = \lambda_i \mathbf{w}_i$$

Eigenvector Properties

- The right and left eigenvectors are orthogonal i.e.

$$\mathbf{w}_i^t \mathbf{v}_i \neq 0 \ , \ \mathbf{w}_i^t \mathbf{v}_j = 0 \quad (i \neq j)$$

- We can normalize the eigenvectors so that:

$$\mathbf{w}_i^t \mathbf{v}_i = 1 \ , \ \mathbf{w}_i^t \mathbf{v}_j = 0 \quad (i \neq j)$$

Eigenvector Example

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}, \ |\mathbf{A} - \lambda \mathbf{I}| = 0 \ \Rightarrow \ \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 3\lambda - 10 = 0 \Rightarrow \ \lambda_{1,2} = \frac{3 \pm \sqrt{(3)^2 + 4(10)}}{2} = \frac{3 \pm \sqrt{49}}{2} = 5, -2$$

Right Eigenvectors $\lambda_1 = 5$

$$\mathbf{A}\mathbf{v}_1 = 5\mathbf{v}_1 \Rightarrow \mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} \Rightarrow \begin{matrix} v_{11} + 4v_{21} = 5v_{11} \\ 3v_{11} + 2v_{21} = 5v_{21} \end{matrix} \quad \textit{choose } \mathbf{v}_{21} = 1 \Rightarrow \mathbf{v}_{11} = 1$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Similarly,

$$\lambda_2 = -2 \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

Eigenvector Example

- Left eigenvectors

$$\lambda_1 = 5 \ \ \mathbf{w}_1^t \mathbf{A} = \mathbf{w}_1^t 5 \Rightarrow [w_{11} \ \ w_{21}] \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} = 5[w_{11} \ \ w_{21}]$$

$$\begin{matrix} w_{11} + 3w_{21} = 5w_{11} \\ 4w_{11} + 2w_{21} = 5w_{21} \end{matrix} \Rightarrow \textit{Let } w_{21} = 4, \textit{ then } \ w_{11} = 3$$

$$\mathbf{w}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \lambda_2 = -2 \Rightarrow \mathbf{w}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ \ \mathbf{v}_2 = \begin{bmatrix} 4 \\ -3 \end{bmatrix} \ \ \mathbf{w}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \ \ \mathbf{w}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\textit{Verify} \ \ \mathbf{w}_1^t\mathbf{v}_1 = 7 \ , \ \ \mathbf{w}_2^t\mathbf{v}_2 = 7 \ , \ \ \mathbf{w}_2^t\mathbf{v}_1 = 0 \ , \ \ \mathbf{w}_1^t\mathbf{v}_2 = 0$$

We would like to make $w_i^t v_i = 1$.
This can be done in many ways.

Eigenvector Example

$$\textit{Let} \quad \mathbf{W} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}$$

$$\textit{Then} \quad \mathbf{W}^T \mathbf{V} = \mathbf{I}$$

$$\textit{Verify} \quad \frac{1}{7}\begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- It can be verified that $\mathbf{W}^T = \mathbf{V}^{-1}$.
- The left and right eigenvectors are used in computing the participation factor matrix.

Modal Matrices

- The deviation away from an equilibrium point can be defined as

$$\Delta\dot{\mathbf{x}} = \mathbf{A}\Delta\mathbf{x}$$

- From this equation it is difficult to determine how parameters in **A** affect a particular **x** because of the variable coupling
- To decouple the problem first define the matrices of the right and left eigenvectors (the modal matrices)

$$\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2 \ldots.. \mathbf{v}_n] \;\; \& \;\; \mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, \ldots.. \mathbf{w}_n]$$

$$\mathbf{AV} = \mathbf{V\Lambda} \quad \text{when} \quad \mathbf{\Lambda} = Diag(\lambda_i)$$

Modal Matrices

- It follows that

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{\Lambda}$$

- To decouple the variables define **z** so

$$\Delta\mathbf{x} = \mathbf{V}\mathbf{z} \rightarrow \Delta\dot{\mathbf{x}} = \mathbf{V}\dot{\mathbf{z}} = \mathbf{A}\Delta\mathbf{x} = \mathbf{A}\mathbf{V}\mathbf{z}$$

- Then

$$\dot{\mathbf{z}} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}\mathbf{z} = \mathbf{W}\mathbf{A}\mathbf{V}\mathbf{z} = \mathbf{\Lambda}\mathbf{z}$$

- Since $\mathbf{\Lambda}$ is diagonal, the equations are now uncoupled with $\dot{z}_i = \lambda_i z_i$

- So $\Delta\mathbf{x}(t) = \mathbf{V}\mathbf{z}(t)$

Example

- Assume the previous system with

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} 1 & 4 \\ 1 & -3 \end{bmatrix}$$

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$$

Modal Matrices

- Thus the response can be written in terms of the individual eigenvalues and right eigenvectors as

$$\Delta\mathbf{x}(t) = \sum_{i=1}^{n} \mathbf{v}_i z_i(0) e^{\lambda_i t}$$

Note, we are requiring that the eigenvalues be distinct!

- Furthermore with

$$\Delta\mathbf{x} = \mathbf{VZ} \Rightarrow \quad \mathbf{z} = \mathbf{V}^{-1}\mathbf{x} = \mathbf{W}^{T}\mathbf{x}$$

- So z(t) can be written as using the left eigenvectors as

$$\mathbf{z}(t) = \mathbf{W}^{t}\mathbf{x}(t) = [\mathbf{w}_1 \ \ \mathbf{w}_2 \ldots \mathbf{w}_n]^{t} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Modal Matrices

- We can then write the response x(t) in terms of the modes of the system

$$z_i(t) = w_i^t x(t)$$

$$z_i(0) = w_i^t x(0) \triangleq c_i$$

$$\text{so} \quad \mathbf{x}(t) = \sum_{i=1}^{n} \mathbf{v}_i c_i e^{\lambda_i t}$$

$$\text{Expanding} \quad \Delta x_i(t) = v_{i1} c_1 e^{\lambda_1 t} + v_{i2} c_2 e^{\lambda_2 t} + \ldots v_{in} c_n e^{\lambda_n t}$$

- So c_i is a scalar that represents the magnitude of excitation of the i^{th} mode from the initial conditions

Numerical example

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}, \ \Delta\mathbf{x}(0) = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

Eigenvalues are $\lambda_1 = -4$, $\lambda_2 = 2$

Eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Modal matrix $\mathbf{V} = \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix}$

Normalize so $\mathbf{V} = \begin{bmatrix} 0.2425 & 0.4472 \\ -0.9701 & 0.8944 \end{bmatrix}$

Numerical example (contd)

Left eigenvector matrix is:

$$\mathbf{W}^{\mathrm{T}} = \mathbf{V}^{-1} = \begin{bmatrix} 1.3745 & -0.6872 \\ 1.4908 & 0.3727 \end{bmatrix}$$

$$\dot{\mathbf{z}} = \mathbf{W}^{\mathrm{T}}\mathbf{A}\mathbf{V}\mathbf{z}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Numerical example (contd)

$$\dot{z}_1 = -4z_1 \quad , \quad \mathbf{z}(0) = V^{-1}\mathbf{x}(0)$$

$$\dot{z}_2 = 2z_2 \quad , \quad \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} 4.123 \\ 0 \end{bmatrix}$$

$$z_1(t) = z_1(0)e^{-4t} \;\; ; \;\; z_2(t) = z_2(0)e^{2t}, \mathbf{C} = \mathbf{W}^T\mathbf{x}(0) = \begin{bmatrix} 4.123 \\ 0 \end{bmatrix}$$

x = Vz

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

Because of the initial condition, the 2nd mode does not get excited

$$= c_1 \begin{bmatrix} 0.2425 \\ -0.9701 \end{bmatrix} z_1(t) + c_2 \begin{bmatrix} 0.4472 \\ 0.8944 \end{bmatrix} z_2(t) = \sum_{i=1}^{2} c_i \mathbf{v}_i z_i(0) e^{\lambda_i t}$$

Mode Shape, Sensitivity and Participation Factors

- So we have

$$\mathbf{x}(t) = \mathbf{V}\mathbf{z}(t), \quad \mathbf{z}(t) = \mathbf{W}^t\mathbf{x}(t)$$

- **x**(t) are the original state variables, **z**(t) are the transformed variables so that each variable is associated with only <u>one</u> mode.
- From the first equation the right eigenvector gives the "mode shape" i.e. relative activity of state variables when a particular mode is excited.
- For example the degree of activity of state variable x_k in $\mathbf{v}_i$ mode is given by the element V_{ki} of the the right eigenvector matrix **V**

Mode Shape, Sensitivity and Participation Factors

- The magnitude of elements of $\mathbf{v}_i$ give the extent of activities of *n* state variables in the i^{th} mode and angles of elements (if complex) give phase displacements of the state variables with regard to the mode.
- The left eigenvector $\mathbf{w}_i$ identifies which combination of original state variables display only the i^{th} mode.

Eigenvalue Parameter Sensitivity

- To derive the sensitivity of the eigenvalues to the parameters recall $\mathbf{Av}_i = \lambda_i \mathbf{v}_i$; take the partial derivative with respect to A_{kj} by using the chain rule

$$\frac{\partial \mathbf{A}}{\partial \mathbf{A}_{kj}} \mathbf{v}_i + \mathbf{A} \frac{\partial \mathbf{v_i}}{\partial A_{kj}} = \frac{\partial \lambda_i}{\partial A_{kj}} \mathbf{v}_i + \lambda_i \frac{\partial \mathbf{v}_i}{\partial A_{kj}}$$

Multiply by $\mathbf{w}_i^t$

$$\mathbf{w}_i^t \frac{\partial \mathbf{A}}{\partial \mathbf{A}_{kj}} \mathbf{v}_i + \mathbf{w}_i^t \mathbf{A} \frac{\partial \mathbf{v_i}}{\partial A_{kj}} = \mathbf{w}_i^t \frac{\partial \lambda_i}{\partial A_{kj}} \mathbf{v}_i + \mathbf{w}_i^t \lambda_i \frac{\partial \mathbf{v}_i}{\partial A_{kj}}$$

$$\mathbf{w}_i^t \frac{\partial \mathbf{A}}{\partial A_{kj}} \mathbf{v}_i + \mathbf{w}_i^t [\mathbf{A} - \lambda_i \mathbf{I}] \frac{\partial \mathbf{v}_i}{\partial A_{kj}} = \mathbf{w}_i^t \frac{\partial \lambda_i}{\partial A_{kj}} \mathbf{v}_i$$

Eigenvalue Parameter Sensitivity

- This is simplified by noting that $\mathbf{w}_i^t(\mathbf{A} - \lambda_i \mathbf{I}) = 0$ by the definition of $\mathbf{w}_i$ being a left eigenvector
- Therefore

$$\mathbf{w}_i^t \frac{\partial \mathbf{A}}{\partial A_{kj}} \mathbf{v}_i = \frac{\partial \lambda_i}{\partial A_{kj}}$$

- Since all elements of $\frac{\partial \mathbf{A}}{\partial A_{kj}}$ are zero, except the k^{th} row, j^{th} column is 1
- Thus $\frac{\partial \lambda_i}{\partial A_{kj}} = W_{ki} V_{ji}$

Sensitivity Example

- In the previous example we had

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}, \quad \lambda_{1,2} = 5, -2, \quad \mathbf{V} = \begin{bmatrix} 1 & 4 \\ 1 & -3 \end{bmatrix}, \quad \mathbf{W} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}$$

- Then the sensitivity of λ_1 and λ_2 to changes in **A** are

$$\frac{\partial \lambda_i}{\partial A_{kj}} = W_{ki} V_{ji} \rightarrow \frac{\partial \lambda_1}{\partial \mathbf{A}} = \frac{1}{7}\begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}, \quad \frac{\partial \lambda_2}{\partial \mathbf{A}} = \frac{1}{7}\begin{bmatrix} 4 & -3 \\ -4 & 3 \end{bmatrix}$$

- For example with

$$\hat{\mathbf{A}} = \begin{bmatrix} 1 & 4 \\ 3 & 3 \end{bmatrix}, \quad \hat{\lambda}_{1,2} = 5.61, -1.61,$$

Eigenvalue Parameter Sensitivity

- This is simplified by noting that $\mathbf{w}_i^t(\mathbf{A}-\lambda_i\mathbf{I})=0$ by the definition of $\mathbf{w}_i$ being a left eigenvector
- Therefore

$$\mathbf{w}_i^t \frac{\partial \mathbf{A}}{\partial A_{kj}} \mathbf{v}_i = \frac{\partial \lambda_i}{\partial A_{kj}}$$

- Since all elements of $\frac{\partial \mathbf{A}}{\partial A_{kj}}$ are zero, except the kth row, jth column is 1
- Thus $\frac{\partial \lambda_i}{\partial A_{kj}} = W_{ki} V_{ji}$

Eigenvalue Parameter Sensitivity

- To derive the sensitivity of the eigenvalues to the parameters recall $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$; take the partial derivative with respect to A_{kj} by using the chain rule

$$\frac{\partial \mathbf{A}}{\partial \mathbf{A}_{kj}} \mathbf{v}_i + \mathbf{A} \frac{\partial \mathbf{v_i}}{\partial A_{kj}} = \frac{\partial \lambda_i}{\partial A_{kj}} \mathbf{v}_i + \lambda_i \frac{\partial \mathbf{v}_i}{\partial A_{kj}}$$

Multiply by $\mathbf{w}_i^t$

$$\mathbf{w}_i^t \frac{\partial \mathbf{A}}{\partial \mathbf{A}_{kj}} \mathbf{v}_i + \mathbf{w}_i^t \mathbf{A} \frac{\partial \mathbf{v_i}}{\partial A_{kj}} = \mathbf{w}_i^t \frac{\partial \lambda_i}{\partial A_{kj}} \mathbf{v}_i + \mathbf{w}_i^t \lambda_i \frac{\partial \mathbf{v}_i}{\partial A_{kj}}$$

$$\mathbf{w}_i^t \frac{\partial \mathbf{A}}{\partial A_{kj}} \mathbf{v}_i + \mathbf{w}_i^t [\mathbf{A} - \lambda_i \mathbf{I}] \frac{\partial \mathbf{v}_i}{\partial A_{kj}} = \mathbf{w}_i^t \frac{\partial \lambda_i}{\partial A_{kj}} \mathbf{v}_i$$

Eigenvalue Parameter Sensitivity

- This is simplified by noting that $\mathbf{w}_i^t(\mathbf{A}-\lambda_i\mathbf{I})=0$ by the definition of $\mathbf{w}_i$ being a left eigenvector
- Therefore

$$\mathbf{w}_i^t \frac{\partial \mathbf{A}}{\partial A_{kj}} \mathbf{v}_i = \frac{\partial \lambda_i}{\partial A_{kj}}$$

- Since all elements of $\frac{\partial \mathbf{A}}{\partial A_{kj}}$ are zero, except the kth row, jth column is 1
- Thus $\frac{\partial \lambda_i}{\partial A_{kj}} = W_{ki}V_{ji}$

Sensitivity Example

- In the previous example we had

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}, \quad \lambda_{1,2} = 5, -2, \quad \mathbf{V} = \begin{bmatrix} 1 & 4 \\ 1 & -3 \end{bmatrix}, \quad \mathbf{W} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}$$

- Then the sensitivity of λ_1 and λ_2 to changes in **A** are

$$\frac{\partial \lambda_i}{\partial A_{kj}} = W_{ki} V_{ji} \rightarrow \frac{\partial \lambda_1}{\partial \mathbf{A}} = \frac{1}{7}\begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}, \quad \frac{\partial \lambda_2}{\partial \mathbf{A}} = \frac{1}{7}\begin{bmatrix} 4 & -3 \\ -4 & 3 \end{bmatrix}$$

- For example with $\hat{\mathbf{A}} = \begin{bmatrix} 1 & 4 \\ 3 & 2.1 \end{bmatrix}, \quad \hat{\lambda}_{1,2} = 5.057, -1.957$

Participation Factors

- The participation factors, P_{ki}, are used to determine how much the k^{th} state variable participates in the i^{th} mode

$$P_{ki} = V_{ki} W_{ki}$$

- The sum of the participation factors for any mode or any variable sum to 1
- The participation factors are quite useful in relating the eigenvalues to portions of a model

Participation Factors

- For the previous example with $P_{ki} = V_{ki}W_{ik}$ and

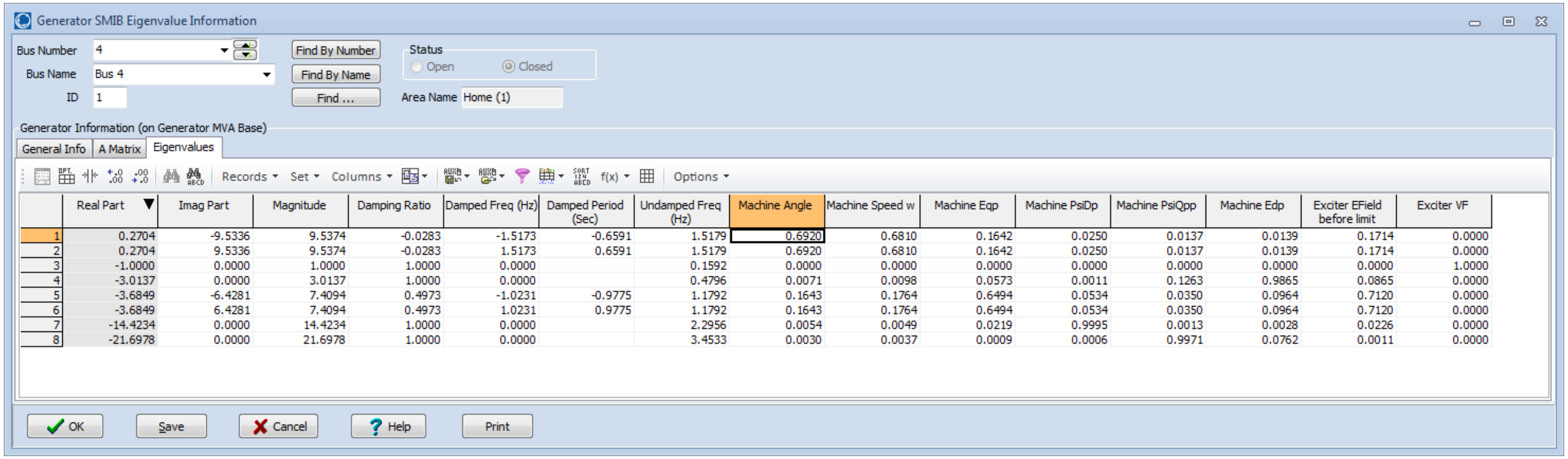
$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} 1 & 4 \\ 1 & -3 \end{bmatrix}, \quad \mathbf{W} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}$$

- We get

$$\mathbf{P} = \frac{1}{7}\begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$$

PowerWorld SMIB Participation Factors

- The magnitudes of the participation factors are shown on the PowerWorld SMIB dialog
- The below values are shown for the four bus example with $Q_4 = 0$



	Real Part	Imag Part	Magnitude	Damping Ratio	Damped Freq (Hz)	Damped Period (Sec)	Undamped Freq (Hz)	Machine Angle	Machine Speed w	Machine Eqp	Machine PsiDp	Machine PsiQpp	Machine Edp	Exciter EField before limit	Exciter VF
1	0.2704	-9.5336	9.5374	-0.0283	-1.5173	-0.6591	1.5179	0.6920	0.6810	0.1642	0.0250	0.0137	0.0139	0.1714	0.0000
2	0.2704	9.5336	9.5374	-0.0283	1.5173	0.6591	1.5179	0.6920	0.6810	0.1642	0.0250	0.0137	0.0139	0.1714	0.0000
3	-1.0000	0.0000	1.0000	1.0000	0.0000		0.1592	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
4	-3.0137	0.0000	3.0137	1.0000	0.0000		0.4796	0.0071	0.0098	0.0573	0.0011	0.1263	0.9865	0.0865	0.0000
5	-3.6849	-6.4281	7.4094	0.4973	-1.0231	-0.9775	1.1792	0.1643	0.1764	0.6494	0.0534	0.0350	0.0964	0.7120	0.0000
6	-3.6849	6.4281	7.4094	0.4973	1.0231	0.9775	1.1792	0.1643	0.1764	0.6494	0.0534	0.0350	0.0964	0.7120	0.0000
7	-14.4234	0.0000	14.4234	1.0000	0.0000		2.2956	0.0054	0.0049	0.0219	0.9995	0.0013	0.0028	0.0226	0.0000
8	-21.6978	0.0000	21.6978	1.0000	0.0000		3.4533	0.0030	0.0037	0.0009	0.0006	0.9971	0.0762	0.0011	0.0000

Case is saved as B4_GENROU_Sat_SMIB_QZero

Measurement Based Modal Analysis

- With the advent of large numbers of PMUs, measurement based SSA is increasing used
  - The goal is to determine the damping associated with the dominant oscillatory modes in the system
  - Approaches seek to approximate a sampled signal by a series of exponential functions (usually damped sinusoidals)
- Several techniques are available with Prony analysis the oldest
  - Method, which was developed by Gaspard Riche de Prony, dates to 1795; power system applications from about 1980's
- Here we'll consider a newer alternative, based on the variable projection method

Some Useful References

- J.F. Hauer, C.J. Demeure, and L.L. Scharf, "Initial results in Prony analysis of power system response signals," *IEEE Trans. Power Systems*, vol.5, pp 80-89, Feb 1990
- D.J. Trudnowski, J.M. Johnson, and J.F. Hauer, "Making Prony analysis more accurate using multiple signals," *IEEE Trans. Power Systems*, vol.14, pp.226-231, Feb 1999
- A. Borden, B.C. Lesieutre, J. Gronquist, "Power System Modal Analysis Tool Developed for Industry Use," *Proc. 2013 North American Power Symposium*, Manhattan, KS, Sept. 2013

Variable Projection Method (VPM)

- Idea of all techniques is to approximate a signal, $y_{org}(t)$, by the sum of other, simpler signals (basis functions)
  - Basis functions are usually exponentials, with linear and quadratic functions also added to detrend the signal
  - Properties of the original signal can be quantified from basis function properties (such as frequency and damping)
  - Signal is considered over an interval with t=0 at the beginning
- Approaches work by sampling the original signal $y_{org}(t)$
- Vector **y** consists of m uniformly sampled points from $y_{org}(t)$ at a sampling value of ΔT, starting with t=0, with values y_j for j=1…m
  - Times are then $t_j = (j-1)\Delta T$

Variable Projection Method (VPM)

- At each time point j, where $t_j = (j-1)\Delta T$ the approximation of y_j is

$$\hat{y}_j(\boldsymbol{\alpha}) = \sum_{i=1}^{n} b_i \phi_i(t_j, \boldsymbol{\alpha})$$

where $\boldsymbol{\alpha}$ is a vector with the real and imaginary eigenvalue components, with $\phi_i(t_j, \boldsymbol{\alpha}) = e^{\alpha_i t_j}$ for a_i corresponding to a real eigenvalue, and $\phi_i(t_j, \boldsymbol{\alpha}) = e^{\alpha_i t_j} \cos(\alpha_{i+1} t_j)$ and $\phi_{i+1}(\boldsymbol{\alpha}) = e^{\alpha_i t_j} \sin(\alpha_{i+1} t_j)$ for a complex eigenvector value

Variable Projection Method (VPM)

- Error (residual) value at each point j is

$$r_j(t_j,\boldsymbol{\alpha}) = y_j - \hat{y}_j(t_j,\boldsymbol{\alpha})$$

  - $\boldsymbol{\alpha}$ is the vector containing the optimization variables

- Function being minimized is

$$\frac{1}{2}\sum_{j=1}^{m}(y_j - \hat{y}_j(t_j,\boldsymbol{\alpha}))^2 = \frac{1}{2}\left\|\mathbf{r}(\boldsymbol{\alpha})\right\|_2^2$$

r(α) is the residual vector

Method iteratively changes α to reduce the minimization function

Variable Projection Method (VPM)

- A key insight of the variable projection method is that

  $\hat{\mathbf{y}}(\boldsymbol{\alpha}) = \boldsymbol{\Phi}(\boldsymbol{\alpha})\mathbf{b}$

  And then the residual is minimized by selecting

  $\mathbf{b} = \boldsymbol{\Phi}(\boldsymbol{\alpha})^{+}\mathbf{y}$

  where $\boldsymbol{\Phi}(\boldsymbol{\alpha})$ is the m by n matrix with values

  $\Phi_{ji}(\boldsymbol{\alpha}) = e^{\alpha_i t_j}$ if α_{i} corresponds to a real eigenvalue,

  and $\Phi_{ji}(\boldsymbol{\alpha}) = e^{\alpha_i t_j}\cos(\alpha_{i+1}t_j)$ and $\Phi_{ji+1}(\boldsymbol{\alpha}) = e^{\alpha_i t_j}\sin(\alpha_{i+1}t_j)$

  for a complex eigenvalue; $t_j = \left(j-1\right)\Delta T$

  Finally, $\boldsymbol{\Phi}(\boldsymbol{\alpha})^{+}$ is the pseudoinverse of $\boldsymbol{\Phi}(\boldsymbol{\alpha})$