ECEN 615
Methods of Electric Power Systems Analysis

Lecture 8: Advanced Power Flow, Gaussian Elimination, Sparse Systems

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Announcements

- Read Chapter 7 from the book
- Homework 2 is due on Thursday September 26

Bus Branch versus Node Breaker

- Due to a variety of issues during the 1970's and 1980's the real-time operations and planning stages of power systems adopted different modeling approaches

Real-Time Operations

Use detailed node/breaker model

EMS system as a set of integrated applications and processes

Real-time operating system

Real-time databases

Planning

Use simplified bus/branch model

PC approach

Use of files

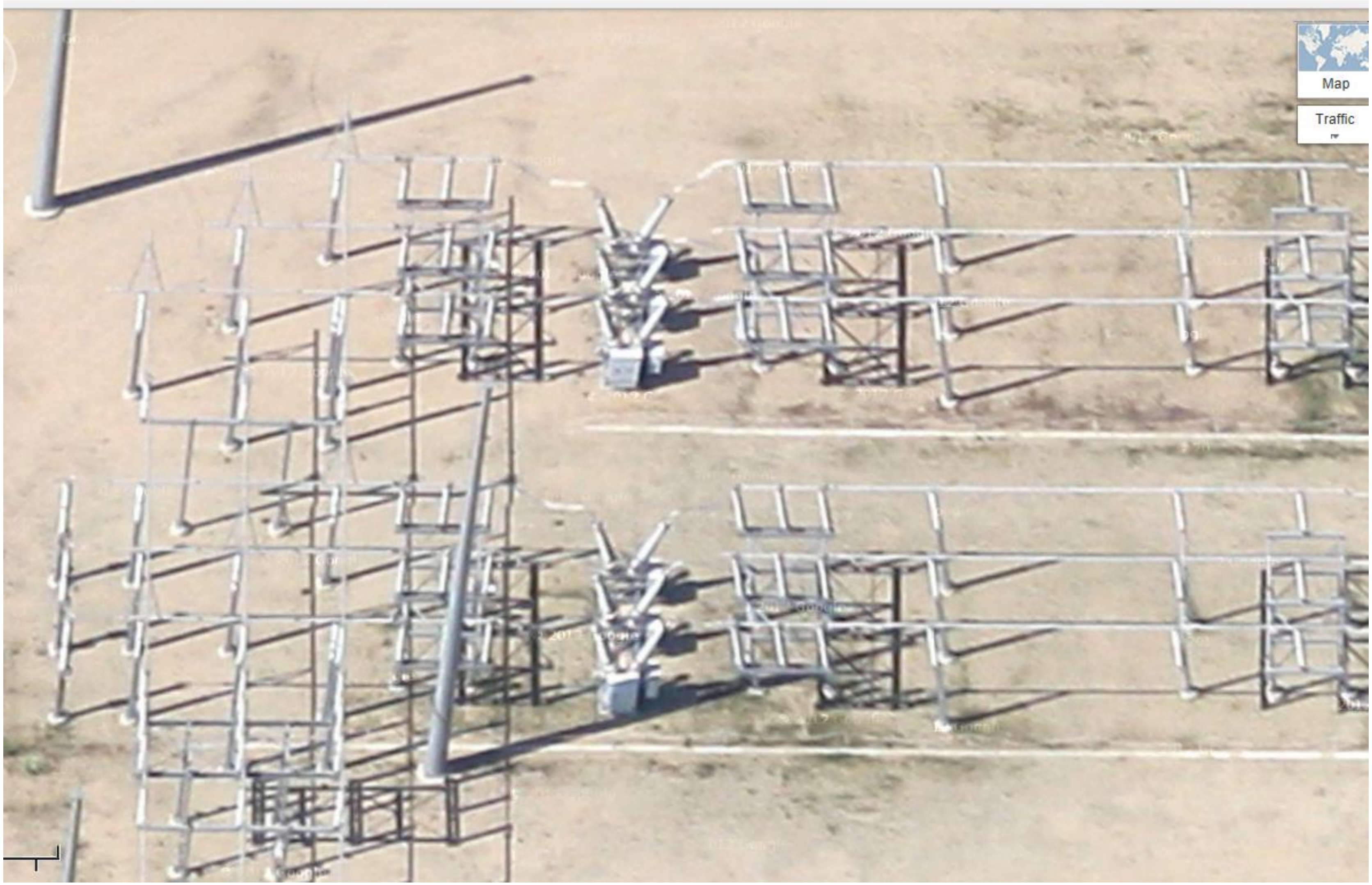
Stand-alone applications

Entire data sets and software tools developed around these two distinct power system models

Circuit Breakers and Disconnects

- Circuit breakers are devices that are designed to clear fault current, which can be many times normal operating current
  - AC circuit breakers take advantage of the current going through zero twice per cycle
  - Transmission faults can usually be cleared in less than three cycles
- Disconnects cannot clear fault current, and usually not normal current. They provide a visual indication the line is open. Can be manual or motorized.
- In the power flow they have essentially no impedance; concept of a zero branch reactance (ZBR)

Google View of a 345 kV Substation



Substation Configurations

- Several different substation breaker/disconnect configurations are common:
- Single bus: simple but a fault any where requires taking out the entire substation; also doing breaker or disconnect maintenance requires taking out the associated line

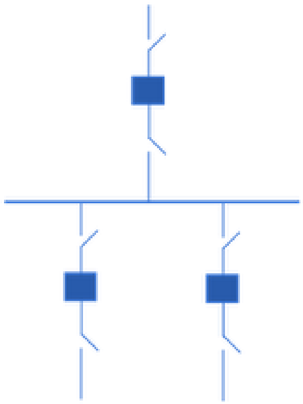


Fig B: Single Bus

Source: http://www.skm-eleksys.com/2011/09/substation-bus-schemes.html

Substation Configurations, cont.

- Main and Transfer Bus:
  Now the breakers can be taken out for maintenance without taking out a line, but protection is more difficult, and a fault on one line will take out at least two

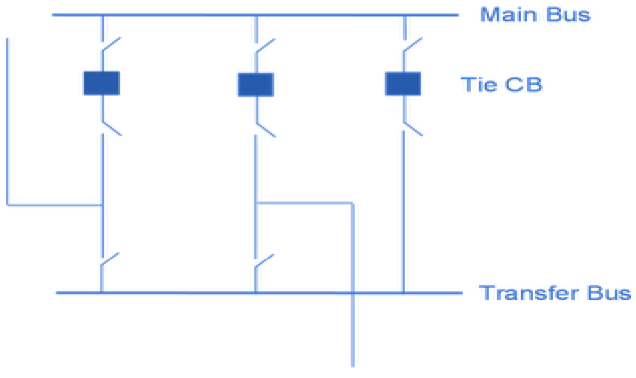


Fig C: Main and Transfer Bus

- Double Bus Breaker:
  Now each line is fully protected when a breaker is out, so high reliability, but more costly

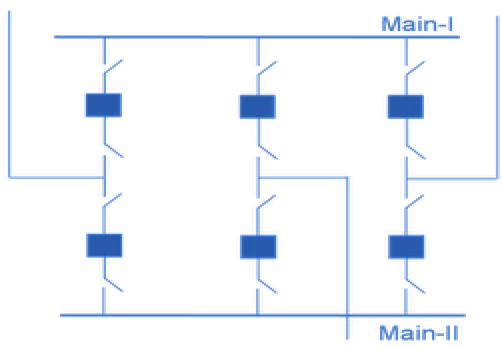


Fig D: Double Bus Double Breaker

Source: http://www.skm-eleksys.com/2011/09/substation-bus-schemes.html

Ring Bus, Breaker and Half

- As the name implies with a ring bus the breakers form a ring; number of breakers is same as number of devices; any breaker can be removed for maintenance
- The breaker and half has two buses and uses three breakers for two devices; both breakers and buses can be removed for maintenance

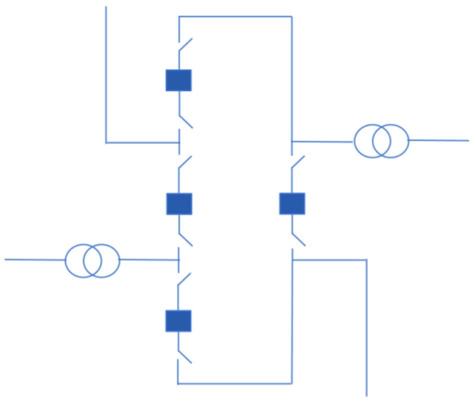


Fig F: Ring Bus

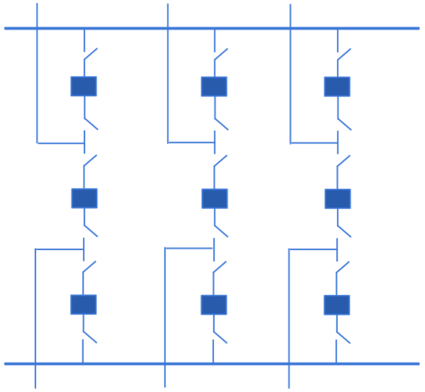
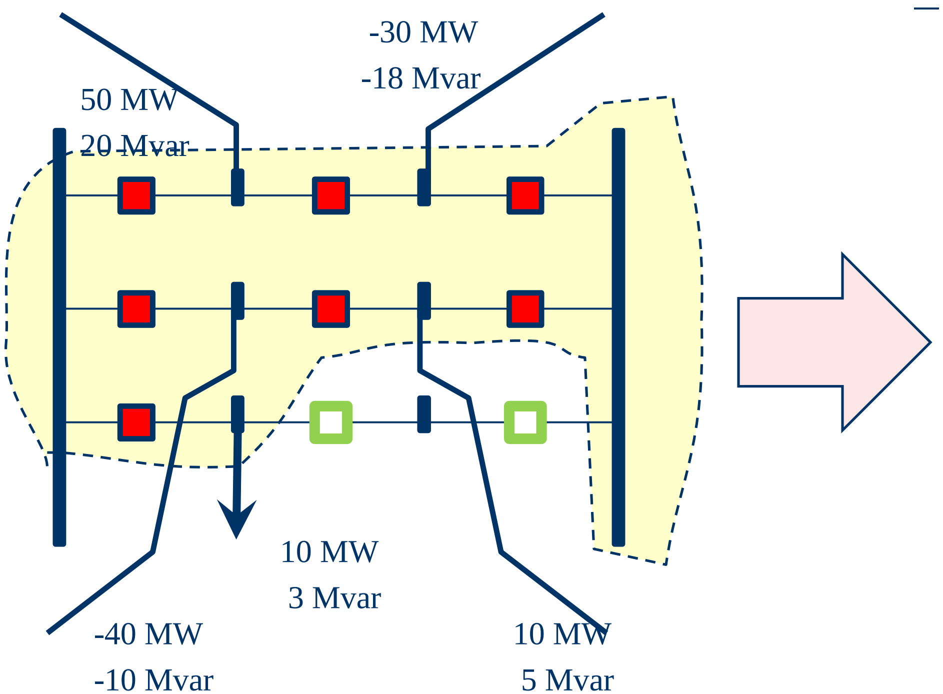


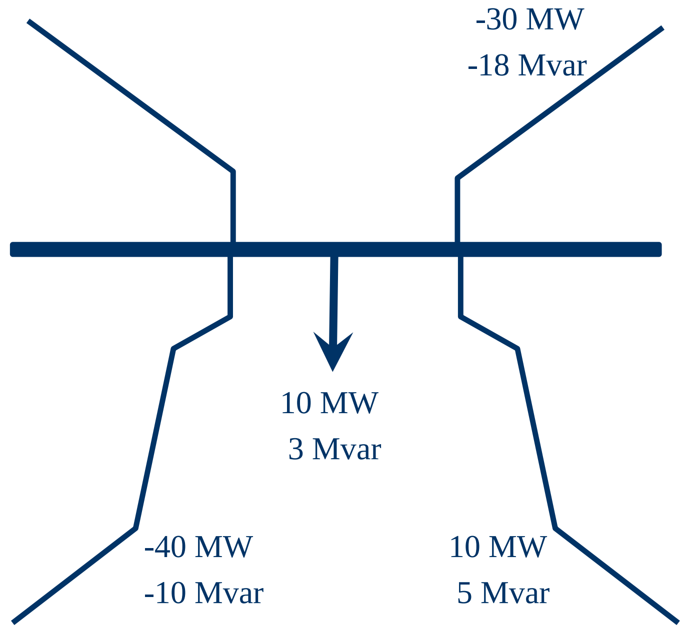
Fig G: Breaker and Half

Source: http://www.skm-eleksys.com/2011/09/substation-bus-schemes.html

EMS and Planning Models

- EMS Model
  - Used for real-time operations
  - Called full topology model
  - Has node-breaker detail
- Planning Model
  - Used for off-line analysis
  - Called consolidated model by PowerWorld
  - Has bus/branch detail

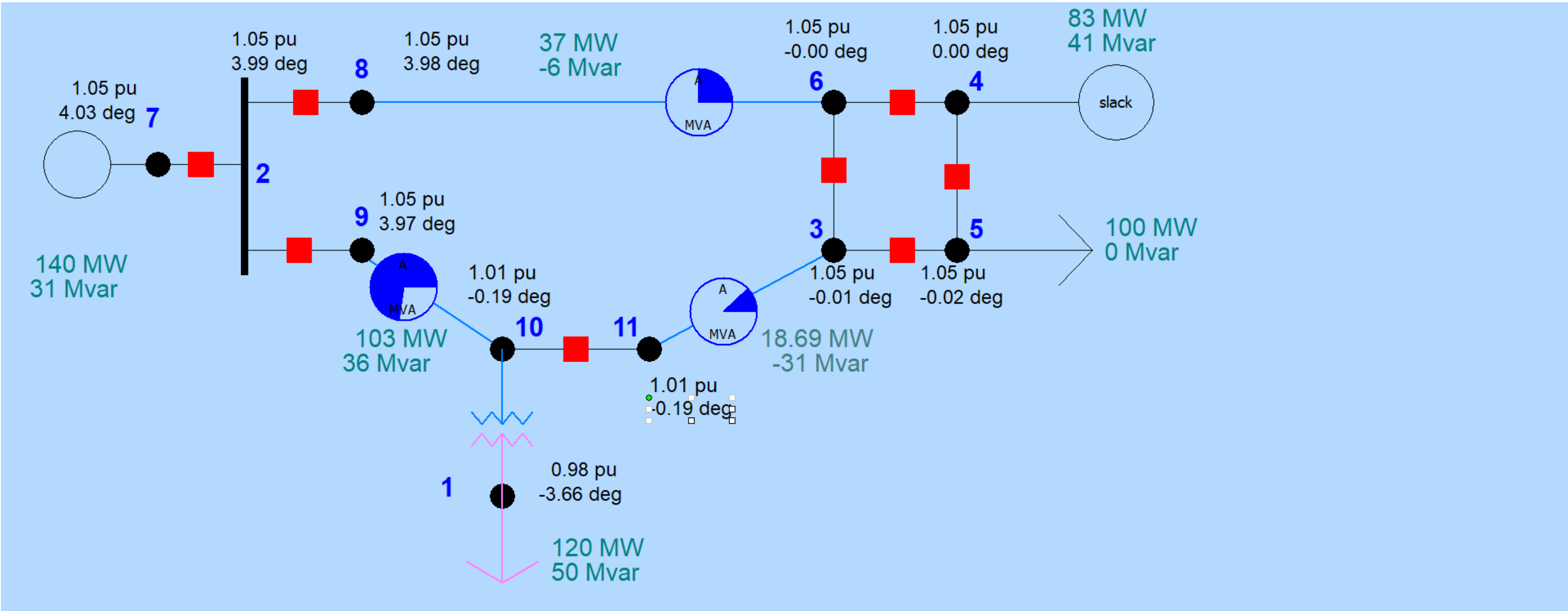




Node-Breaker Consolidation

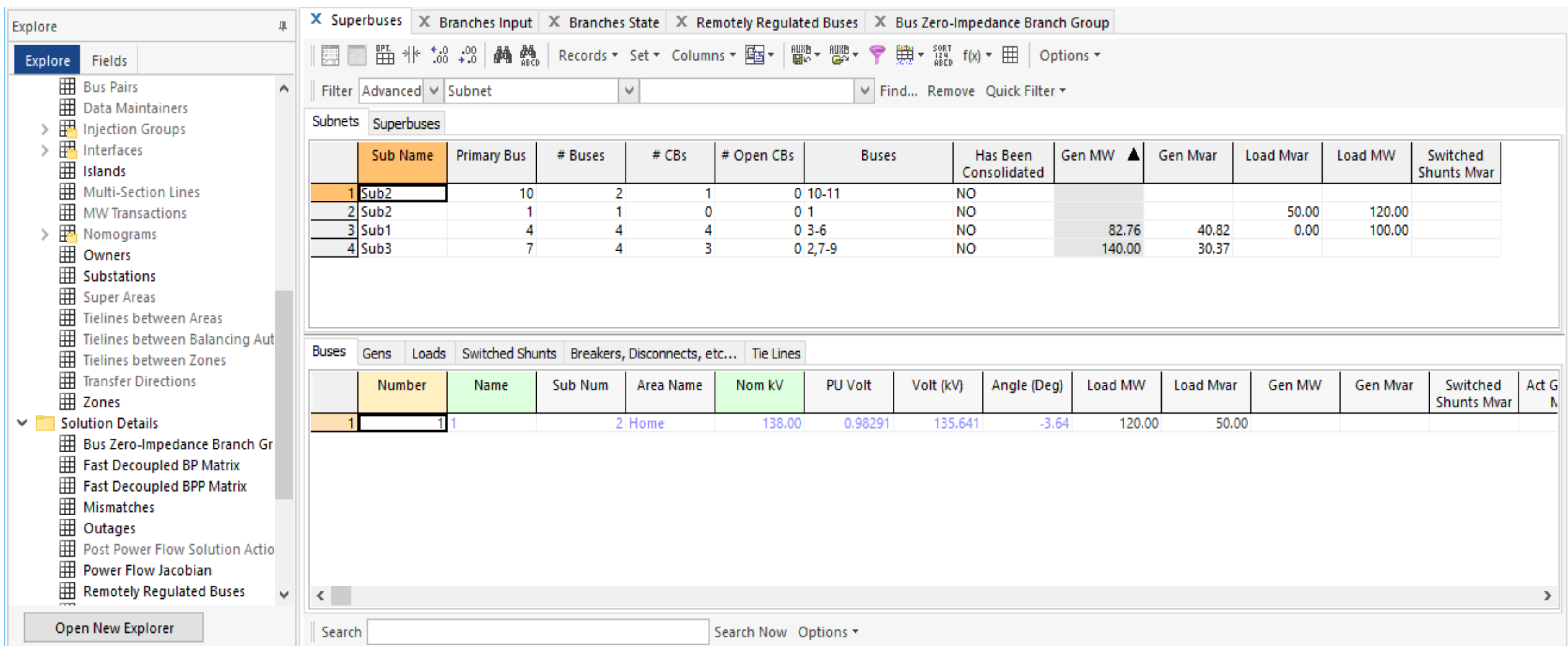
- One approach to modeling systems with large numbers of ZBRs (zero branch reactances, such as from circuit breakers) is to just assume a small reactance and solve
  - This results in lots of buses and branches, resulting in a much larger problem
  - This can cause numerical problems in the solution
- The alterative is to consolidate the nodes that are connected by ZBRs into a smaller number of buses
  - After solution all nodes have the same voltage; use logic to determine the device flows

Node-Breaker Example



Case name is **FT_11Node**. PowerWorld consolidates nodes (buses) into super buses; available in the Model Explorer: Solution, Details, Superbuses.

Node-Breaker Example



Note there is ambiguity on how much power is flowing in each device in the ring bus (assuming each device really has essentially no impedance)

Linear System Solution: Introduction

- A problem that occurs in many is fields is the solution of linear systems **Ax = b** where **A** is an n by n matrix with elements a_{ij}, and **x** and **b** are n-vectors with elements x_i and b_i respectively
- In power systems we are particularly interested in systems when n is relatively large and **A** is sparse
  - How large is large is changing
- A matrix is sparse if a large percentage of its elements have zero values
- Goal is to understand the computational issues (including complexity) associated with the solution of these systems

Introduction, cont.

- Sparse matrices arise in many areas, and can have domain specific structures
  - Symmetric matrices
  - Structurally symmetric matrices
  - Tridiagnonal matrices
  - Banded matrices
- A good (and free) book on sparse matrices is available at www-users.cs.umn.edu/~saad/IterMethBook_2ndEd.pdf
- ECEN 615 is focused on problems in the electric power domain; it is not a general sparse matrix course
  - Much of the early sparse matrix work was done in power!

Gaussian Elimination

- The best known and most widely used method for solving linear systems of algebraic equations is attributed to Gauss
- Gaussian elimination avoids having to explicitly determine the inverse of **A**, which is $O(n^3)$
- Gaussian elimination can be readily applied to sparse matrices
- Gaussian elimination leverages the fact that scaling a linear equation does not change its solution, nor does adding on linear equation to another

$$2x_1 + 4x_2 = 10 \rightarrow x_1 + 2x_2 = 5$$

Gaussian Elimination, cont.

- Gaussian elimination is the elementary procedure in which we use the first equation to eliminate the first variable from the last n-1 equations, then we use the new second equation to eliminate the second variable from the last n-2 equations, and so on
- After performing n-1 such eliminations we end up with a triangular system which is easily solved in a backward direction

Example 1

- We need to solve for **x** in the system

$$\begin{bmatrix} 2 & 3 & -1 & 0 \\ -6 & -5 & 0 & 2 \\ 2 & -5 & 6 & -6 \\ 4 & 2 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 20 \\ -45 \\ -3 \\ 30 \end{bmatrix}$$

- The three elimination steps are given on the next slides; for simplicity, we have appended the r.h.s. vector to the matrix
- First step is set the diagonal element of row 1 to 1 (i.e., normalize it)

Example 1, cont.

- Eliminate x_1 by subtracting row 1 from all the rows below it

$$\begin{array}{l} \text{multiply row } 1 \text{ by } \frac{1}{2} \\ \text{multiply row } 1 \text{ by } 6 \\ \text{and add to row } 2 \\ \text{multiply row } 1 \text{ by } -2 \\ \text{and add to row } 3 \\ \text{multiply row } 1 \text{ by } -4 \\ \text{and add to row } 4 \end{array} \quad \left[\begin{array}{cccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & 0 & 10 \\ 0 & 4 & -3 & \frac{1}{2} & 15 \\ 0 & -8 & -7 & -6 & -23 \\ 0 & -4 & 7 & -3 & -10 \end{array}\right]$$

Example 1, cont.

- Eliminate x_2 by subtracting row 2 from all the rows below it

multiply row 2 by $\frac{1}{4}$

multiply row 2 by 8

and add to row 3

multiply row 2 by 4

and add to row 4

$$\left[\begin{array}{cccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & 0 & 10 \\ 0 & 1 & -\frac{3}{4} & \frac{1}{2} & \frac{15}{4} \\ 0 & 0 & 1 & -2 & 7 \\ 0 & 0 & 1 & -1 & 5 \end{array}\right]$$

Example 1, cont.

- Elimination of x_3 from row 3 and 4

subtract row 3

from row 4

$$\left[\begin{array}{cccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & 0 & 10 \\ 0 & 1 & -\frac{3}{4} & \frac{1}{2} & \frac{15}{4} \\ 0 & 0 & 1 & -2 & 7 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

Example 1, cont.

- Then, we solve for x by "going backwards", i.e., using back substitution:

$$x_4 = -2$$

$$x_3 - 2x_4 = 7 \Rightarrow x_3 = 3$$

$$x_2 - \frac{3}{4}x_3 + \frac{1}{2}x_4 = \frac{15}{4} \Rightarrow x_2 = 7$$

$$x_1 + \frac{3}{2}x_2 - \frac{1}{2}x_3 = 10 \Rightarrow x_1 = 1$$

Triangular Decomposition

- In this example, we have:
  - triangularized the original matrix by Gaussian elimination using column elimination
  - then, we used back substitution to solve the triangularized system
- The following slides present a general scheme for triangular decomposition by Gaussian elimination
- The assumption is that **A** is a nonsingular matrix (hence its inverse exists)
- Gaussian elimination also requires the diagonals to be nonzero; this can be achieved through ordering
- If **b** is zero then we have a trivial solution **x** = **0**

Triangular Decomposition

- We form the matrix $\mathbf{A}_a$ using $\mathbf{A}$ and $\mathbf{b}$ with

$$\mathbf{A}_a = \left[\mathbf{A} \vdots \mathbf{b}\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{bmatrix}$$

and show the steps of the triangularization scheme

Triangular Decomposition, Step 1

- Step 1: normalize the first equation

$$a_{1j}^{(1)} = \frac{a_{1j}}{a_{11}} \qquad j = 2, \ldots, n$$

$$b_1^{(1)} = \frac{b_1}{a_{11}}$$

Triangular Decomposition, Step 2

- Step 2: a) eliminate x_1 from row 2:

$$a_{2j}^{(1)} = a_{2j} - a_{21} a_{1j}^{(1)}, \quad j = 2, \ldots, n$$

$$b_2^{(1)} = b_2 - a_{21} b_1^{(1)}$$

- Step 2: b) normalize the second equation

$$a_{2j}^{(2)} = \frac{a_{2j}^{(1)}}{a_{22}^{(1)}}, \quad j = 3, \ldots, n$$

$$b_2^{(2)} = \frac{b_2^{(1)}}{a_{22}^{(1)}}$$

Triangular Decomposition, Step 2

and we end up at the end of step 2 with

$$\begin{bmatrix} 1 & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} & b_1^{(1)} \\ 0 & 1 & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} & b_2^{(2)} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & b_n \end{bmatrix}$$

Triangular Decomposition, Step 3

- Step 3: a) eliminate x_1 and x_2 from row 3:

$$a_{3j}^{(1)} = a_{3j} - a_{31} a_{1j}^{(1)} \qquad j = 2, \ldots, n$$

$$b_3^{(1)} = b_3 - a_{31} b_1^{(1)}$$

$$a_{3j}^{(2)} = a_{3j}^{(1)} - a_{32}^{(1)} a_{2j}^{(2)} \qquad j = 3, \ldots, n$$

$$b_3^{(2)} = b_3^{(1)} - a_{32}^{(1)} b_2^{(2)}$$

- Step 3: b) normalize the third equation:

$$a_{3j}^{(3)} = \frac{a_{3j}^{(2)}}{a_{33}^{(2)}} \qquad j = 4, \ldots, n$$

$$b_3^{(3)} = \frac{b_3^{(2)}}{a_{33}^{(2)}}$$

Triangular Decomposition, Step 3

and we have the system at the end of step 3

$$
\begin{bmatrix}
1 & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & \cdots & a_{1n}^{(1)} & b_{1}^{(1)} \\
0 & 1 & a_{21}^{(2)} & a_{24}^{(2)} & \cdots & a_{2n}^{(2)} & b_{2}^{(2)} \\
0 & 0 & 1 & a_{34}^{(3)} & \cdots & a_{3n}^{(3)} & b_{3}^{(3)} \\
a_{41} & a_{42} & a_{43} & a_{44} & \cdots & a_{4n} & b_{4} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nn} & b_{n}
\end{bmatrix}
$$

Triangular Decomposition, Step k

- In general, we have for step k:

  a) eliminate $x_1, x_2, \ldots, x_{k-1}$ from row k:

$$a_{kj}^{(m)} = a_{kj}^{(m-1)} - a_{km}^{(m-1)} a_{mj}^{(m)} \qquad j = m+1, \ldots, n$$

$$b_{k}^{(m)} = b_{k}^{(m-1)} - a_{km}^{(m-1)} b_{m}^{(m)} \qquad m = 1, 2, \ldots, k-1$$

  b) normalize the k[th] equation:

$$a_{kj}^{(k)} = \frac{a_{kj}^{(k-1)}}{a_{kk}^{(k-1)}} \qquad j = k+1, \ldots, n$$

$$b_{k}^{(k)} = \frac{b_{k}^{(k-1)}}{a_{kk}^{(k-1)}}$$

Triangular Decomposition: Upper Triangular Matrix

- and proceed in this manner until we obtain the upper triangular matrix (the n[th] derived system):

$$\begin{bmatrix} 1 & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & \cdots & a_{1n}^{(1)} & b_{1}^{(1)} \\ 0 & 1 & a_{21}^{(2)} & a_{24}^{(2)} & \cdots & a_{2n}^{(2)} & b_{2}^{(2)} \\ 0 & 0 & 1 & a_{34}^{(3)} & \cdots & a_{3n}^{(3)} & b_{3}^{(3)} \\ 0 & 0 & 0 & 1 & \cdots & a_{4n}^{(4)} & b_{4}^{(4)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & b_{n}^{(n)} \end{bmatrix}$$

Triangular Decomposition

- Note that in the scheme presented, unlike in the first example, we triangularly decompose the system by eliminating row-wise rather than column-wise
  - In successive rows we eliminate (reduce to *0*) each element to the left of the diagonal rather than those below the diagonal
  - Either could be used, but row-wise operations will work better for power system sparse matrices

Solving for X

- To compute x we perform back substitution

$$x_n = b_n^{(n)}$$

$$x_{n-1} = b_{n-1}^{(n-1)} - a_{n-1,n}^{(n-1)} x_n$$

$$\vdots$$

$$x_k = b_k^{(k)} - \sum_{j=k+1}^{n} a_{kj}^{(k)} x_j \qquad k = n-1,\ n-2,\ \ldots,\ 1$$

Upper Triangular Matrix

- The triangular decomposition scheme applied to the matrix **A** results in the upper triangular matrix **U** with the elements

$$
u_{ij} = \begin{cases} 1 & i = j \\ a_{ij}^{(i)} & j > i \\ 0 & j < i \end{cases}
$$

- The following theorem is important in the development of the sparse computational scheme

LU Decomposition Theorem

- Any nonsingular matrix **A** has the following factorization:

$$\mathbf{A} = \mathbf{LU}$$

where **U** could be the upper triangular matrix previously developed (with 1's on its diagonals) and **L** is a lower triangular matrix defined by

$$\ell_{ij} = \begin{cases} a_{ij}^{(j-1)} & j \le i \\ 0 & j > i \end{cases}$$

LU Decomposition Application

- As a result of this theorem we can rewrite

  $$\mathbf{Ax} = \mathbf{LUx} = \mathbf{b}$$

  Define $\mathbf{y} = \mathbf{Ux}$

  Then $\mathbf{Ly} = \mathbf{b}$
- Can also be set so **U** has non unity diagonals
- Once **A** has been factored, we can solve for **x** by first solving for **y**, a process known as forward substitution, then solving for **x** in a process known as back substitution
- In the previous example we can think of **L** as a record of the forward operations preformed on **b**.

LDU Decomposition

- In the previous case we required that the diagonals of **U** be unity, while there was no such restriction on the diagonals of **L**
- An alternative decomposition is

$$\mathbf{A} = \tilde{\mathbf{L}}\mathbf{D}\mathbf{U}$$

$$\text{with } \mathbf{L} = \tilde{\mathbf{L}}\mathbf{D}$$

where **D** is a diagonal matrix, and the lower triangular matrix is modified to require unity for the diagonals

Symmetric Matrix Factorization

- The LDU formulation is quite useful for the case of a symmetric matrix

$$\mathbf{A} = \mathbf{A}^T$$

$$\mathbf{A} = \tilde{\mathbf{L}}\mathbf{D}\mathbf{U} = \mathbf{U}^T\mathbf{D}\tilde{\mathbf{L}}^T = \mathbf{A}^T$$

$$\mathbf{U} = \tilde{\mathbf{L}}^T$$

$$\mathbf{A} = \mathbf{U}^T\mathbf{D}\mathbf{U}$$

- Hence only the upper triangular elements and the diagonal elements need to be stored, reducing storage by almost a factor of 2

Symmetric Matrix Factorization

- There are also some computational benefits from factoring symmetric matrices. However, since symmetric matrices are not common in power applications, we will not consider them in-depth
- However, topologically symmetric sparse matrices are quite common, so those will be our main focus

Pivoting

- An immediate problem that can occur with Gaussian elimination is the issue of zeros on the diagonal; for example

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

- This problem can be solved by a process known as "pivoting," which involves the interchange of either both rows and columns (full pivoting) or just the rows (partial pivoting)
  - Partial pivoting is much easier to implement, and actually can be shown to work quite well

Pivoting, cont.

- In the previous example the (partial) pivot would just be to interchange the two rows

$$\tilde{\mathbf{A}} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

  obviously we need to keep track of the interchanged rows!
- Partial pivoting can be helpful in improving numerical stability even when the diagonals are not zero
  - When factoring row k interchange rows so the new diagonal is the largest element in column k for rows j >= k

LU Algorithm Without Pivoting Processing by row

- We will use the more common approach of having ones on the diagonals of **L**. Also in the common, diagonally dominant power system problems pivoting is not needed Below algorithm is in row form (useful with sparsity!)

```
For i := 2 to n Do Begin  // This is the row being processed
  For j := 1 to i-1 Do Begin  // Rows subtracted from row i
    A[i,j] = A[i,j]/A[j,j]  // This is the scaling
    For k := j+1 to n Do Begin  // Go through each column in i
      A[i,k] = A[i,k] - A[i,j]*A[j,k]
    End;
  End;
End;
```

LU Example

- Starting matrix

$$\mathbf{A} = \begin{bmatrix} 20 & -12 & -5 \\ -5 & 12 & -6 \\ -4 & -3 & 8 \end{bmatrix}$$

- First row is unchanged; start with i=2
- Result with i=2, j=1; done with row 2

$$\mathbf{A} = \begin{bmatrix} 20 & -12 & -5 \\ -0.25 & 9 & -7.25 \\ -4 & -3 & 8 \end{bmatrix}$$

LU Example, cont.

- Result with i=3, j=1;

$$\mathbf{A} = \begin{bmatrix} 20 & -12 & -5 \\ -0.25 & 9 & -7.25 \\ -0.2 & -5.4 & 7 \end{bmatrix}$$

- Result with i=3, j=2; done with row 3; done!

$$\mathbf{A} = \begin{bmatrix} 20 & -12 & -5 \\ -0.25 & 9 & -7.25 \\ -0.2 & -0.6 & 2.65 \end{bmatrix}$$

LU Example, cont.

- Original matrix is used to hold **L** and **U**

$$\mathbf{A} = \begin{bmatrix} 20 & -12 & -5 \\ -0.25 & 9 & -7.25 \\ -0.2 & -0.6 & 2.65 \end{bmatrix} = \mathbf{LU}$$

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ -0.2 & -0.6 & 1 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 20 & -12 & -5 \\ 0 & 9 & -7.25 \\ 0 & 0 & 2.65 \end{bmatrix}$$

With this approach the original **A** matrix has been replaced by the factored values!

Forward Substitution

Forward substitution solves **b = Ly** with values in **b** being over written (replaced by the **y** values)

```
For i := 2 to n Do Begin  // This is the row being processed
  For j := 1 to i-1 Do Begin
    b[i] = b[i] - A[i,j]*b[j]    // This is just using the L matrix
  End;
End;
```

Forward Substitution Example

$$\text{Let } \mathbf{b} = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

$$\text{From before } \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ -0.2 & -0.6 & 1 \end{bmatrix}$$

$$y[1] = 10$$

$$y[2] = 20 - (-0.25)*10 = 22.5$$

$$y[3] = 30 - (-0.2)*10 - (-0.6)*22.5 = 45.5$$

Backward Substitution

- Backward substitution solves $\mathbf{y} = \mathbf{U}\mathbf{x}$ (with values of **y** contained in the **b** vector as a result of the forward substitution)

```
For i := n to 1 Do Begin  // This is the row being processed
  For j := i+1 to n Do Begin
    b[i] = b[i] - A[i,j]*b[j]    // This is just using the U matrix
  End;
  b[i] = b[i]/A[i,i]    // The A[i,i] values are <> 0 if it is nonsingular
End
```

Backward Substitution Example

$$\text{Let } \mathbf{y} = \begin{bmatrix} 10 \\ 22.5 \\ 45.5 \end{bmatrix}$$

$$\text{From before } \mathbf{U} = \begin{bmatrix} 20 & -12 & -5 \\ 0 & 9 & -7.25 \\ 0 & 0 & 2.65 \end{bmatrix}$$

$$x[3] = (1/2.65) * 45.5 = 17.17$$

$$x[2] = (1/9) * \left(22.5 - (-7.25) * 17.17\right) = 16.33$$

$$x[1] = (1/20) * \left(10 - (-5) * 17.17 - (-12) * 16.33\right) = 14.59$$

Computational Complexity

- Computational complexity indicates how the number of numerical operations scales with the size of the problem
- Computational complexity is expressed using the "Big O" notation; assume a problem of size n
  - Adding the number of elements in a vector is O(n)
  - Adding two n by n full matrices is $O(n^2)$
  - Multiplying two n by n full matrices is $O(n^3)$
  - Inverting an n by n full matrix, or doing Gaussian elimination is $O(n^3)$
  - Solving the traveling salesman problem by brute-force search is O(n!)

Computational Complexity

- Knowing the computational complexity of a problem can help to determine whether it can be solved (at least using a particular method)
  - Scaling factors do not affect the computation complexity
    - an algorithm that takes $n^3/2$ operations has the same computational complexity of one the takes $n^3/10$ operations (though obviously the second one is faster!)
- With $O(n^3)$ factoring a full matrix becomes computationally intractable quickly!
  - A 100 by 100 matrix takes a million operations (give or take)
  - A 1000 by 1000 matrix takes a billion operations
  - A 10,000 by 10,000 matrix takes a trillion operations!

Sparse Systems

- The material presented so far applies to any arbitrary linear system
- The next step is to see what happens when we apply triangular factorization to a sparse matrix
- For a sparse system, only nonzero elements need to be stored in the computer since no arithmetic operations are performed on the 0's
- The triangularization scheme is adapted to solve sparse systems in such a way as to preserve the sparsity as much as possible

Sparse Matrix History

- A nice overview of sparse matrix history is by Iain Duff at http://www.siam.org/meetings/la09/talks/duff.pdf
- Sparse matrices developed simultaneously in several different disciplines in the early 1960's with power systems definitely one of the key players (Bill Tinney from BPA)
- Different disciplines claim credit since they didn't necessarily know what was going on in the others

Sparse Matrix History

- In power systems a key N. Sato, W.F. Tinney, “Techniques for Exploiting the Sparsity of the Network Admittance Matrix,” Power App. and Syst., pp 944-950, December 1963
  - In the paper they are proposing solving systems with up to 1000 buses (nodes) in 32K of memory!
  - You’ll also note that in the discussion by El-Abiad, Watson, and Stagg they mention the creation of standard test systems with between 30 and 229 buses (this surely included the now famous 118 bus system)
  - The BPA authors talk “power flow” and the discussors talk “load flow.”
- Tinney and Walker present a much more detailed approach in their 1967 IEEE Proceedings paper titled “Direct Solutions of Sparse Network Equations by Optimally Order Triangular Factorization”

Sparse Matrix Computational Order

- The computational order of factoring a sparse matrix, or doing a forward/backward substitution depends on the matrix structure
  - Full matrix is $O(n^3)$
  - A diagonal matrix is O(n); that is, just invert each element
- For power system problems the classic paper is F. L. Alvarado, "Computational complexity in power systems," *IEEE Transactions on Power Apparatus and Systems*, ,May/June 1976
  - $O(n^{1.4})$ for factoring, $O(n^{1.2})$ for forward/backward
  - For a 100,000 by 100,000 matrix changes computation for factoring from 1 quadrillion to 10 million!

Inverse of a Sparse Matrix

- The inverse of a sparse matrix is NOT in general a sparse matrix
- We never (or at least very, very, very seldom) explicitly invert a sparse matrix
  - Individual columns of the inverse of a sparse matrix can be obtained by solving $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ with $\mathbf{b}$ set to all zeros except for a single nonzero in the position of the desired column
  - If a few desired elements of $\mathbf{A}^{-1}$ are desired (such as the diagonal values) they can usually be computed quite efficiently using sparse vector methods (a topic we'll be considering soon)
- We can't invert a singular matrix (with sparse or not)

Computer Architecture Impacts

- With modern computers the processor speed is many times faster than the time it takes to access data in main memory
  - Some instructions can be processed in parallel
- Caches are used to provide quicker access to more commonly used data
  - Caches are smaller than main memory
  - Different cache levels are used with the quicker caches, like L1, have faster speeds but smaller sizes; L1 might be 64K, whereas the slower L2 might be 1M
- Data structures can have a significant impact on sparse matrix computation