

ECEN 615

Methods of Electric Power Systems Analysis

Lecture 21: Optimal Power Flow, Linear Programming (LP)

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Announcements



- Read Chapter 8 and Appendices 3B and 3E of Chapter 3
- Homeworks 6 and 7 are assigned today, with Homework 6 due on Nov 12 and Homework 7 by Nov 24
- The second exam will be in class on Nov 17
 - Distance learners will be able to take the exam from Nov 16 to Nov 18
- Associated with Homework 7 will be student presentations; these will be about 15 minutes during class on Nov 19 or Nov 24
 - Other times can be arranged for the distance learners

OPF Problem Formulation



- The OPF is usually formulated as a minimization with equality and inequality constraints

Minimize $F(\mathbf{x}, \mathbf{u})$

$$\mathbf{g}(\mathbf{x}, \mathbf{u}) = \mathbf{0}$$

$$\mathbf{h}_{\min} \leq \mathbf{h}(\mathbf{x}, \mathbf{u}) \leq \mathbf{h}_{\max}$$

$$\mathbf{u}_{\min} \leq \mathbf{u} \leq \mathbf{u}_{\max}$$

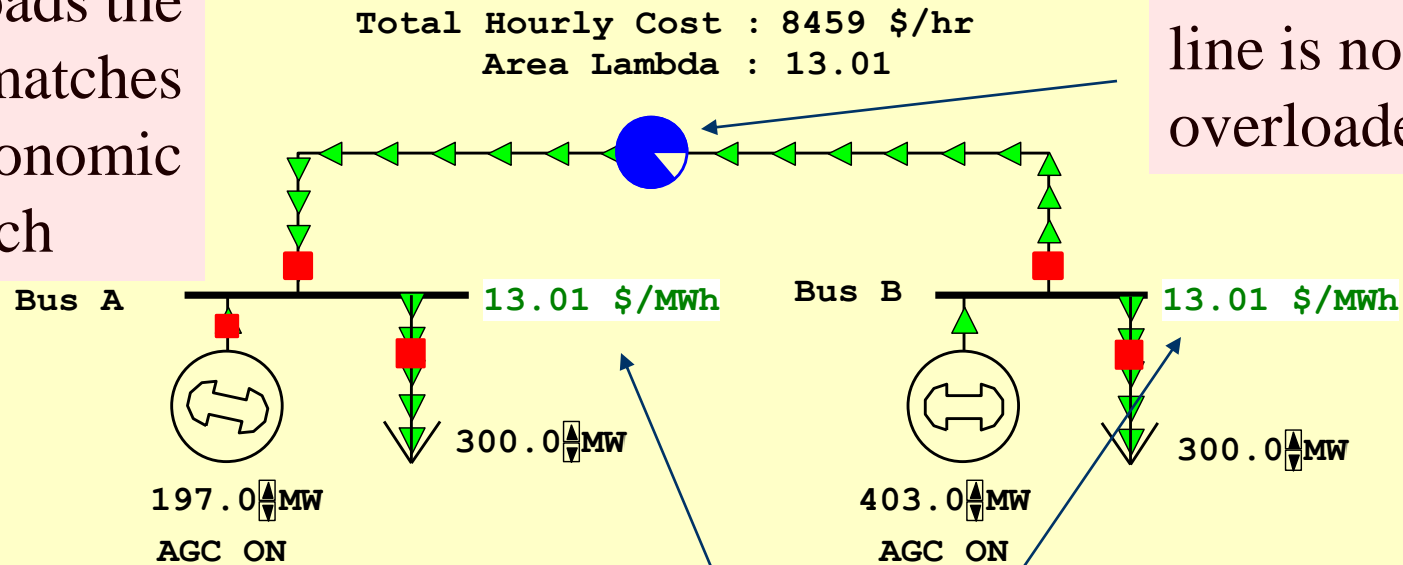
where \mathbf{x} is a vector of dependent variables (such as the bus voltage magnitudes and angles), \mathbf{u} is a vector of the control variables, $F(\mathbf{x}, \mathbf{u})$ is the scalar objective function, \mathbf{g} is a set of equality constraints (e.g., the power balance equations) and \mathbf{h} is a set of inequality constraints (such as line flows)

Two Bus with Unconstrained Line



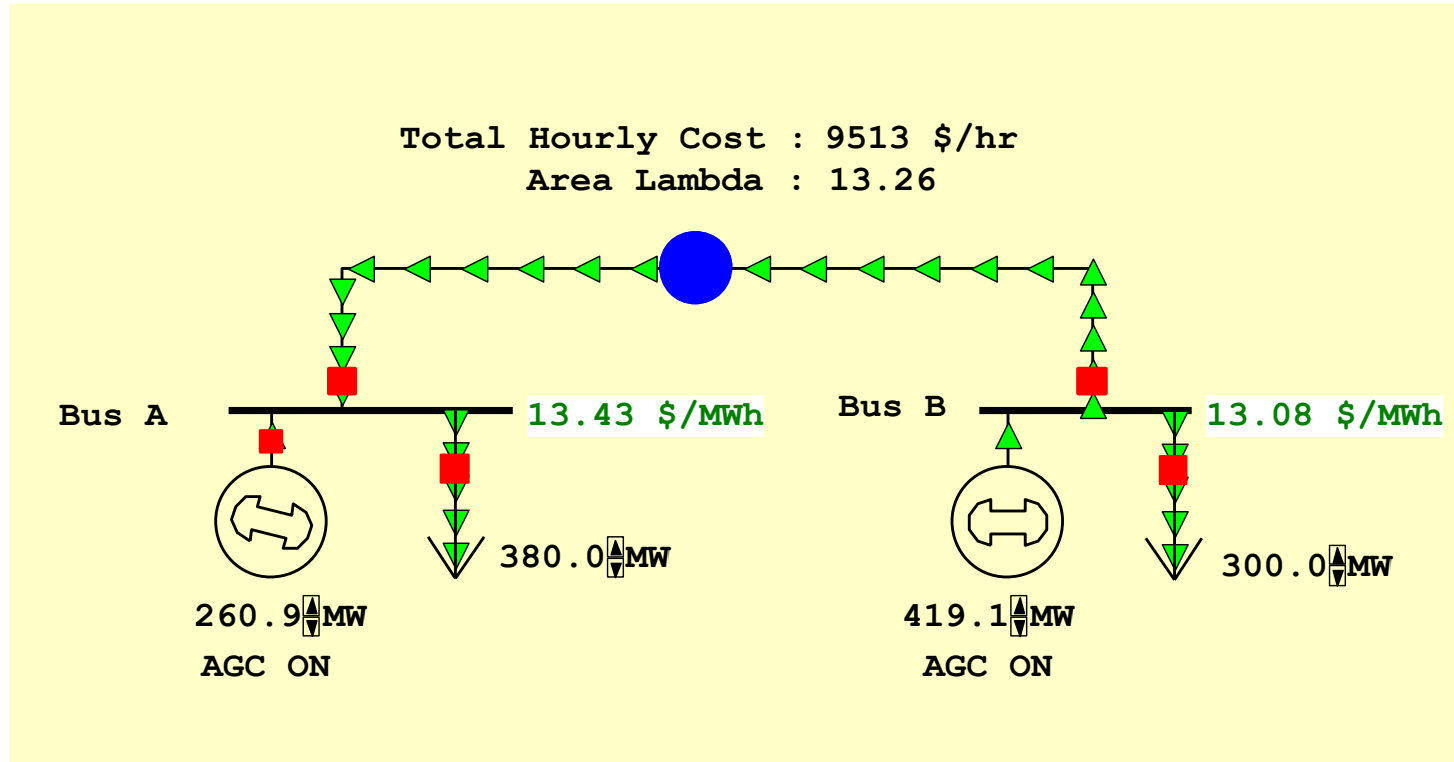
With no overloads the OPF matches the economic dispatch

Transmission line is not overloaded



Marginal cost of supplying power to each bus (locational marginal costs)

Two Bus with Constrained Line



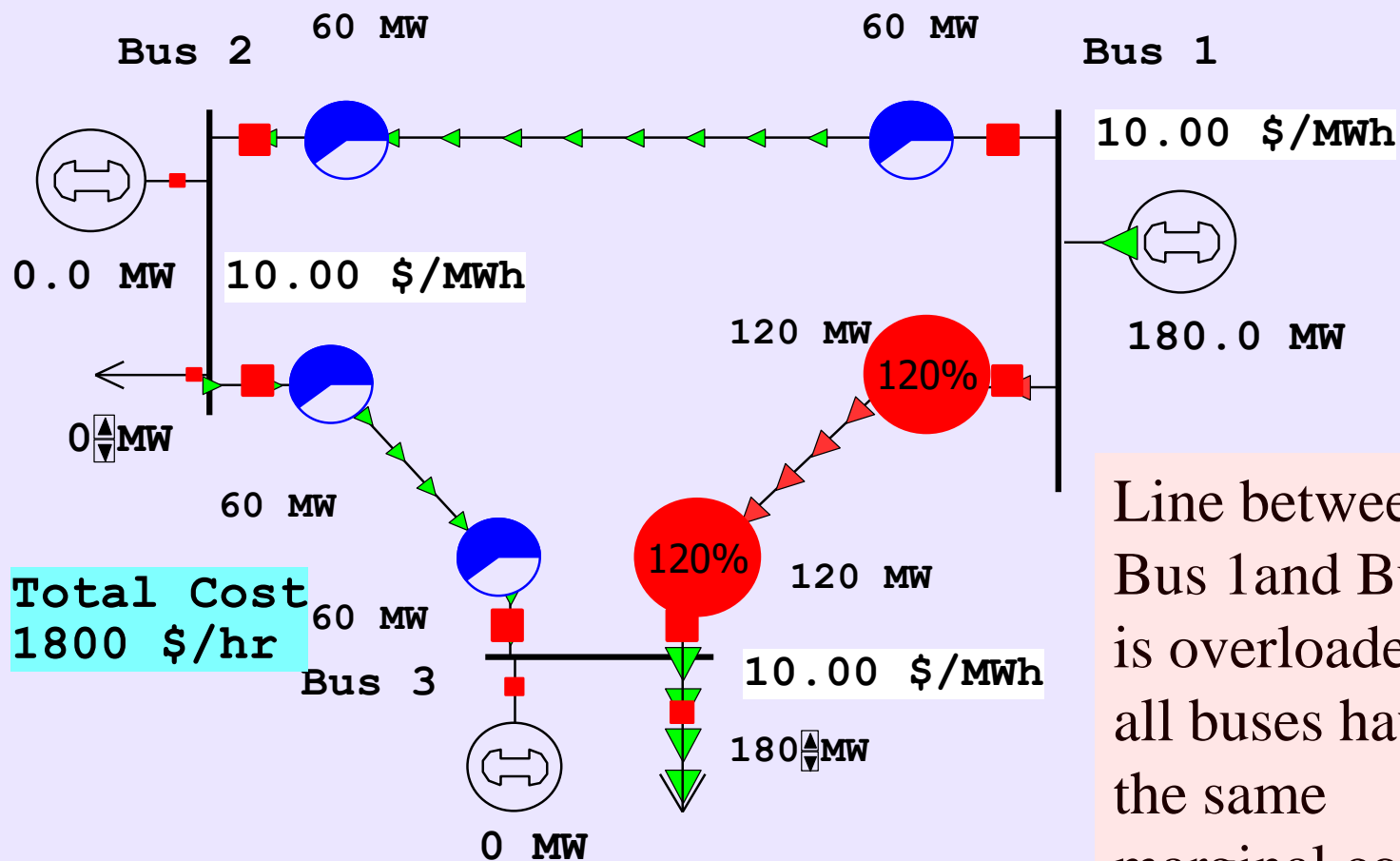
With the line loaded to its limit, additional load at Bus A must be supplied locally, causing the marginal costs to diverge.

Three Bus (B3) Example



- Consider a three bus case (Bus 1 is system slack), with all buses connected through 0.1 pu reactance lines, each with a 100 MVA limit
- Let the generator marginal costs be
 - Bus 1: 10 \$ / MWhr; Range = 0 to 400 MW
 - Bus 2: 12 \$ / MWhr; Range = 0 to 400 MW
 - Bus 3: 20 \$ / MWhr; Range = 0 to 400 MW
- Assume a single 180 MW load at bus 2

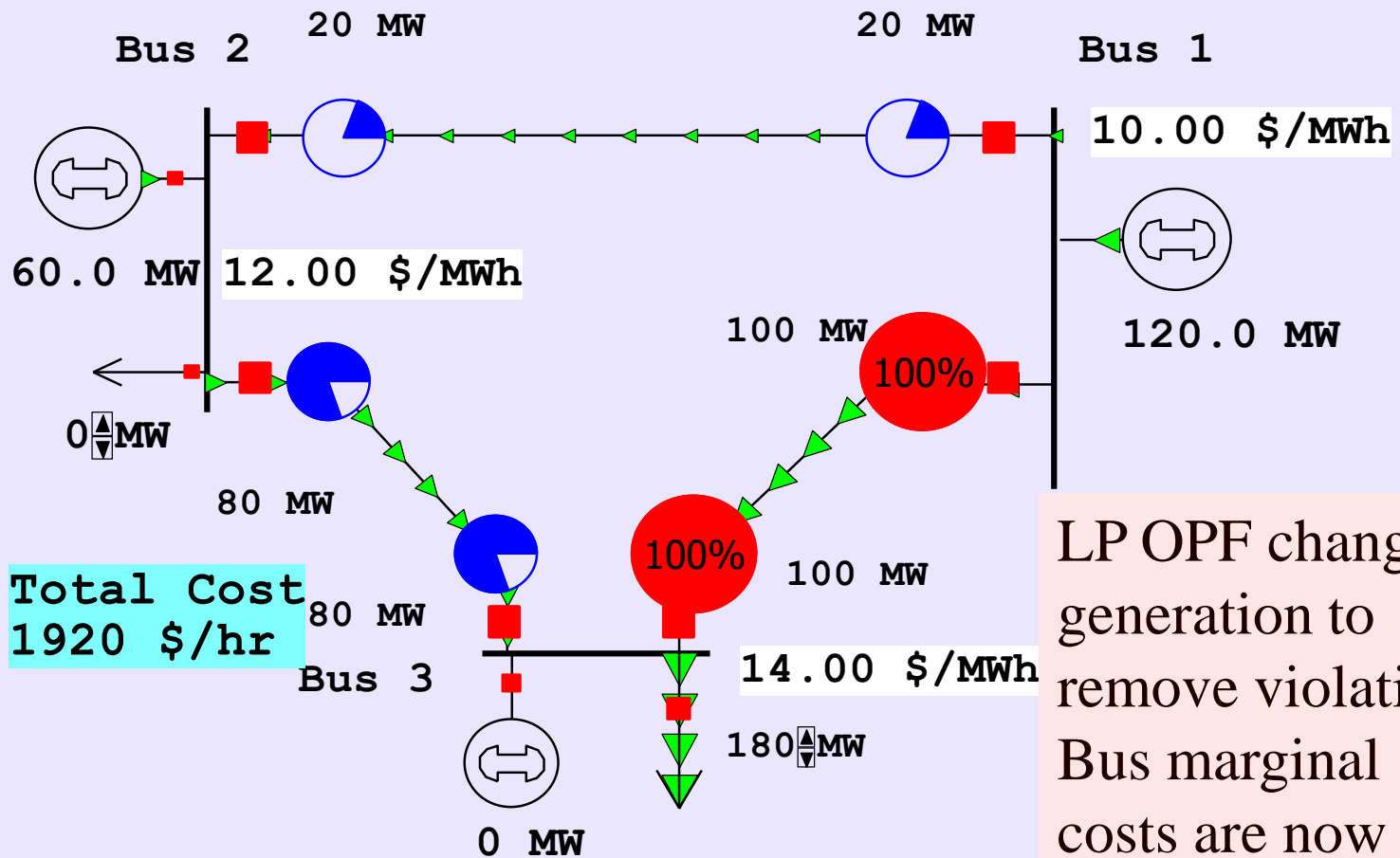
B3 with Line Limits NOT Enforced



Total Cost
1800 \$/hr

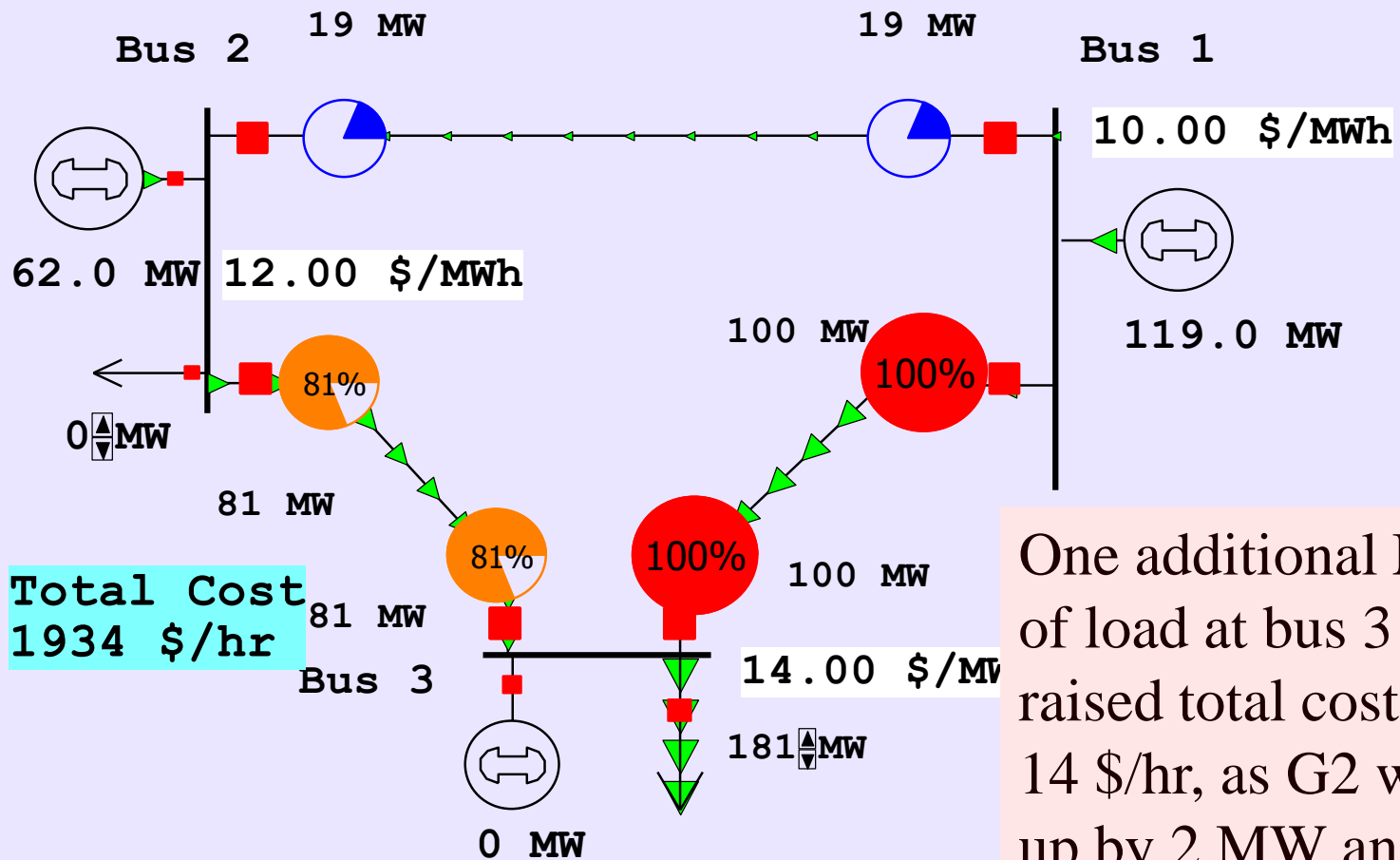
Line between Bus 1 and Bus 3 is overloaded; all buses have the same marginal cost

B3 with Line Limits Enforced



LP OPF changes generation to remove violation. Bus marginal costs are now different.

Verify Bus 3 Marginal Cost



One additional MW of load at bus 3 raised total cost by 14 \$/hr, as G2 went up by 2 MW and G1 went down by 1MW

Why is bus 3 LMP = \$14 /MWh



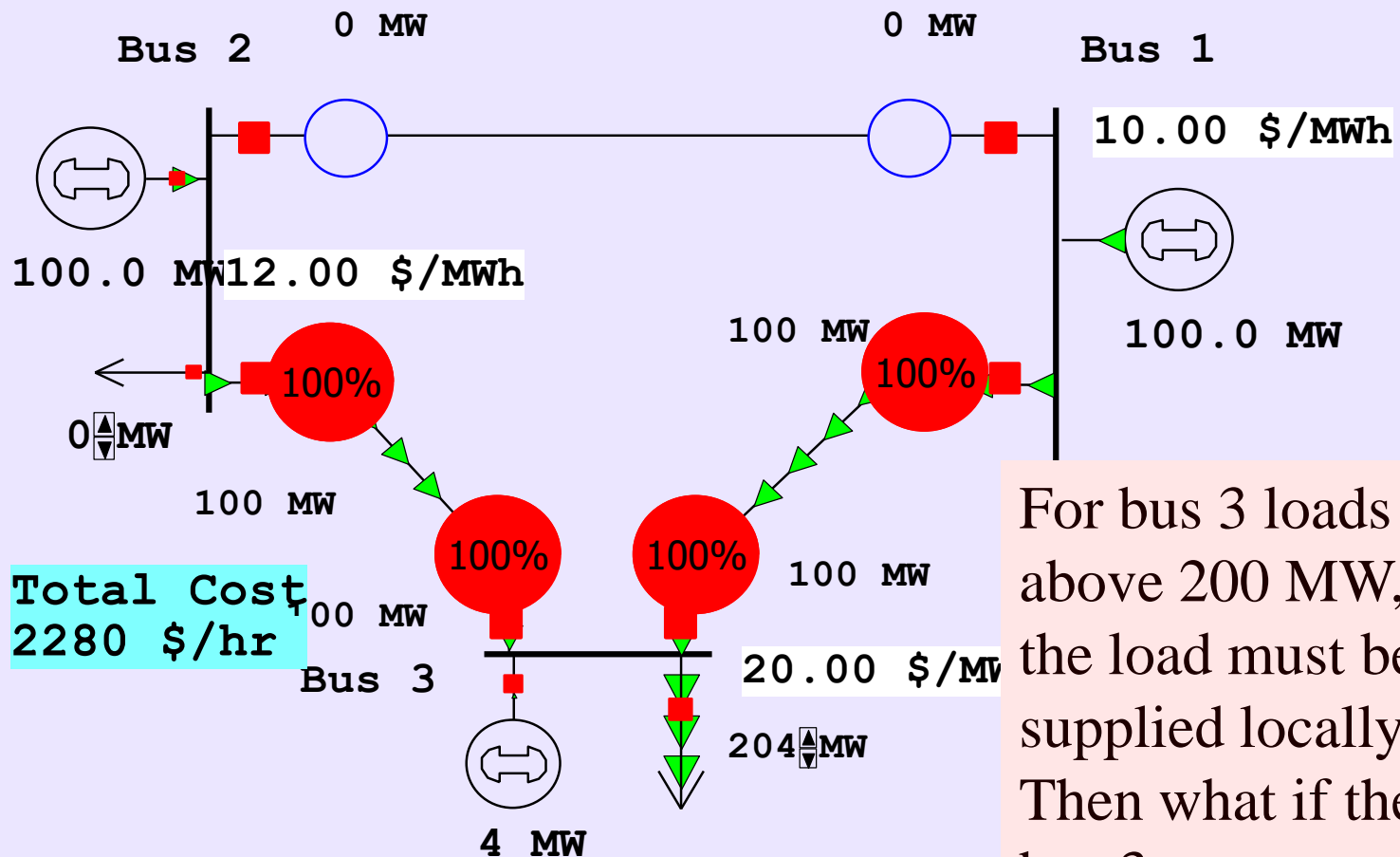
- All lines have equal impedance. Power flow in a simple network distributes inversely to impedance of path.
 - For bus 1 to supply 1 MW to bus 3, $\frac{2}{3}$ MW would take direct path from 1 to 3, while $\frac{1}{3}$ MW would “loop around” from 1 to 2 to 3.
 - Likewise, for bus 2 to supply 1 MW to bus 3, $\frac{2}{3}$ MW would go from 2 to 3, while $\frac{1}{3}$ MW would go from 2 to 1 to 3.

Why is bus 3 LMP \$ 14 / MWh, cont'd



- With the line from 1 to 3 limited, no additional power flows are allowed on it.
- To supply 1 more MW to bus 3 we need
 - $\Delta P_{G1} + \Delta P_{G2} = 1 \text{ MW}$
 - $2/3 \Delta P_{G1} + 1/3 \Delta P_{G2} = 0$; (no more flow on 1-3)
- Solving requires we up P_{G2} by 2 MW and drop P_{G1} by 1 MW -- a net increase of $\$24 - \$10 = \$14$.

Both lines into Bus 3 Congested

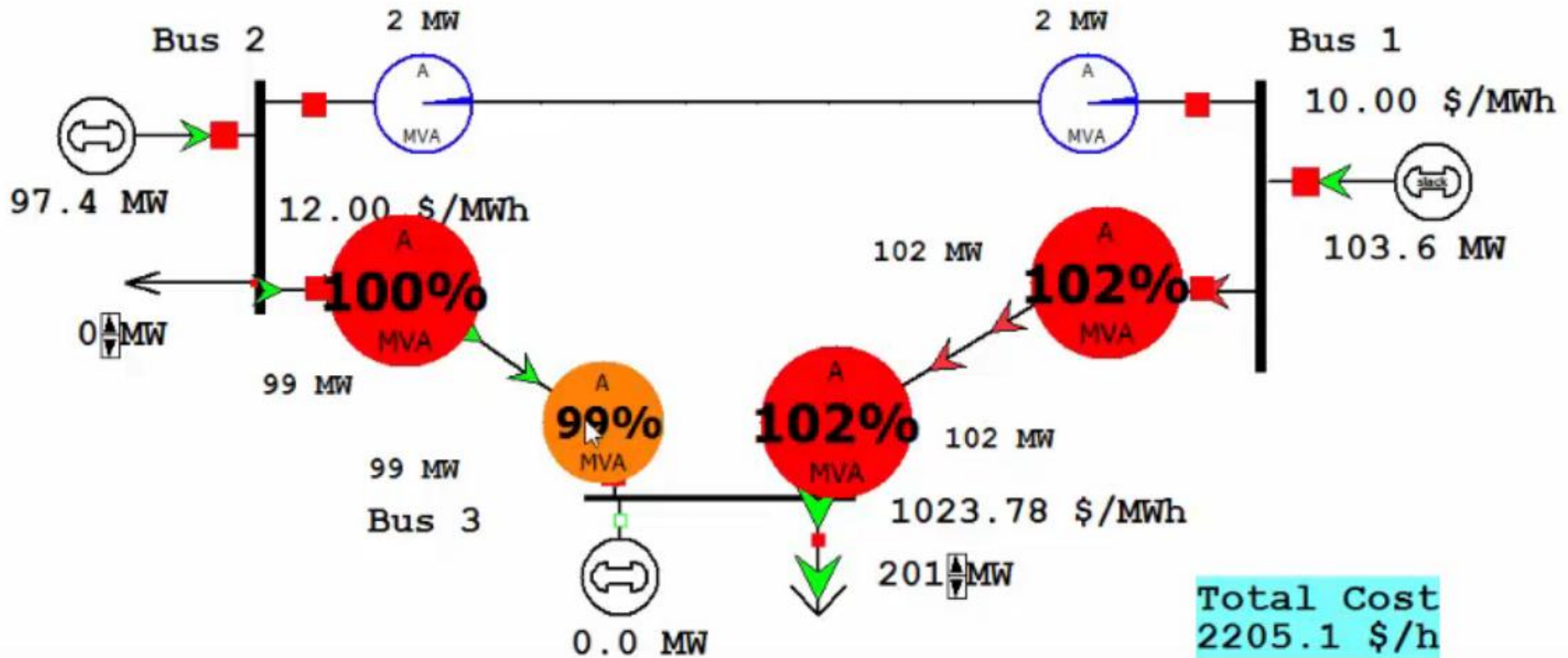


For bus 3 loads above 200 MW, the load must be supplied locally. Then what if the bus 3 generator opens?

Both lines into Bus 3 Congested



Infeasible example



LP OPF Solution Method



- There are different OPF solution techniques. One common approach uses linear programming (LP)
- The LP approach iterates between
 - solving a full ac or dc power flow solution
 - enforces real/reactive power balance at each bus
 - enforces generator reactive limits
 - system controls are assumed fixed
 - takes into account non-linearities
 - solving a primal LP
 - changes system controls to enforce linearized constraints while minimizing cost

Quick Coverage of Linear Programming



- LP is probably the most widely used mathematical programming technique
- It is used to solve linear, constrained minimization (or maximization) problems in which the objective function and the constraints can be written as linear functions

Example Problem 1



- Assume that you operate a lumber mill which makes both construction-grade and finish-grade boards from the logs it receives. Suppose it takes 2 hours to rough-saw and 3 hours to plane each 1000 board feet of construction-grade boards. Finish-grade boards take 2 hours to rough-saw and 5 hours to plane for each 1000 board feet. Assume that the saw is available 8 hours per day, while the plane is available 15 hours per day. If the profit per 1000 board feet is \$100 for construction-grade and \$120 for finish-grade, how many board feet of each should you make per day to maximize your profit?

Problem 1 Setup



Let x_1 = amount of cg, x_2 = amount of fg

$$\text{Maximize } 100x_1 + 120x_2$$

$$\text{s.t. } 2x_1 + 2x_2 \leq 8$$

$$3x_1 + 5x_2 \leq 15$$

$$x_1, x_2 \geq 0$$

Notice that all of the equations are linear, but they are inequality, as opposed to equality, constraints; we are seeking to determine the values of x_1 and x_2

Example Problem 2



- A nutritionist is planning a meal with 2 foods: A and B. Each ounce of A costs \$ 0.20, and has 2 units of fat, 1 of carbohydrate, and 4 of protein. Each ounce of B costs \$0.25, and has 3 units of fat, 3 of carbohydrate, and 3 of protein. Provide the least cost meal which has no more than 20 units of fat, but with at least 12 units of carbohydrates and 24 units of protein.

Problem 2 Setup



Let x_1 = ounces of A, x_2 = ounces of B

Minimize $0.20x_1 + 0.25x_2$

s.t. $2x_1 + 3x_2 \leq 20$

$x_1 + 3x_2 \geq 12$

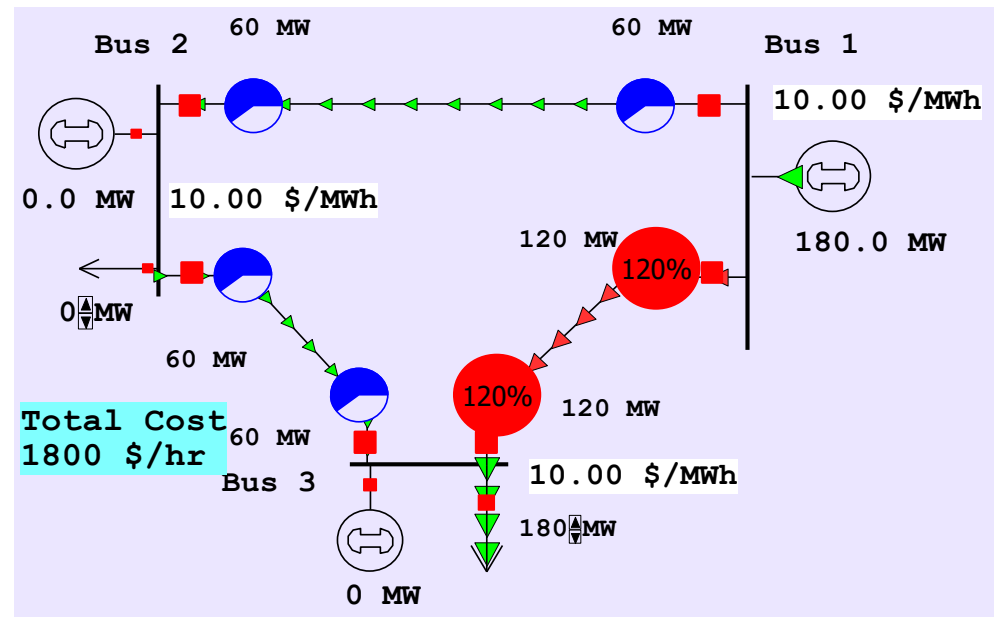
$4x_1 + 3x_2 \geq 24$

Again all of the equations are linear, but they are inequality, as opposed to equality, constraints; we are again seeking to determine the values of x_1 and x_2 ; notice there are also more constraints than solution variables

Three Bus Case Formulation



- For the earlier three bus system given the initial condition of an overloaded transmission line, minimize the cost of generation such that the change in generation is zero, and the flow on the line between buses 1 and 3 is not violating its limit
- Can be setup considering the change in generation, $(\Delta P_{G1}, \Delta P_{G2}, \Delta P_{G3})$



Three Bus Case Problem Setup



Let $x_1 = \Delta P_{G1}$, $x_2 = \Delta P_{G2}$, $x_3 = \Delta P_{G3}$

Minimize $10x_1 + 12x_2 + 20x_3$

s.t. $\frac{2}{3}x_1 + \frac{1}{3}x_2 \leq -20$ Line flow constraint

$x_1 + x_2 + x_3 = 0$ Power balance constraint

enforcing limits on x_1, x_2, x_3

LP Standard Form



The standard form of the LP problem is

Minimize $\mathbf{c}\mathbf{x}$

s.t. $\mathbf{A}\mathbf{x} = \mathbf{b}$

$\mathbf{x} \geq \mathbf{0}$

Maximum problems can be treated as minimizing the negative

where \mathbf{x} = n-dimensional column vector

\mathbf{c} = n-dimensional row vector

\mathbf{b} = m-dimensional column vector

\mathbf{A} = $m \times n$ matrix

For the LP problem usually $n \gg m$

The previous examples were not in this form!

Replacing Inequality Constraints with Equality Constraints



- The LP standard form does not allow inequality constraints
- Inequality constraints can be replaced with equality constraints through the introduction of slack variables, each of which must be greater than or equal to zero

$$K \leq b_i \rightarrow K + y_i = b_i \quad \text{with } y_i \geq 0$$

$$K \geq b_i \rightarrow K - y_i = b_i \quad \text{with } y_i \geq 0$$

- Slack variables have no cost associated with them; they merely tell how far a constraint is from being binding, which will occur when its slack variable is zero

Lumber Mill Example with Slack Variables



- Let the slack variables be x_3 and x_4 , so

$$\text{Minimize } -(100x_1 + 120x_2)$$

Minimize the negative

$$\text{s.t. } 2x_1 + 2x_2 + x_3 = 8$$

$$3x_1 + 5x_2 + x_4 = 15$$

$$x_1, x_2, x_3, x_4 \geq 0$$

LP Definitions



A vector \mathbf{x} is said to be basic if

This is a key LP concept!

1. $\mathbf{Ax} = \mathbf{b}$

2. At most m components of \mathbf{x} are non-zero; these are called the basic variables; the rest are non basic variables; if there are less than m non-zeros then \mathbf{x} is called degenerate

\mathbf{A}_B is called the basis matrix

Define $\mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix}$ (with \mathbf{x}_B basic) and $\mathbf{A} = [\mathbf{A}_B \quad \mathbf{A}_N]$

With $[\mathbf{A}_B \quad \mathbf{A}_N] \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix} = \mathbf{b}$ so $\mathbf{x}_B = \mathbf{A}_B^{-1} (\mathbf{b} - \mathbf{A}_N \mathbf{x}_N)$

Fundamental LP Theorem



- Given an LP in standard form with \mathbf{A} of rank m then
 - If there is a feasible solution, there is a basic feasible solution
 - If there is an optimal, feasible solution, then there is an optimal, basic feasible solution
- Note, there could be a LARGE number of basic, feasible solutions
 - Simplex algorithm determines the optimal, basic feasible solution usually very quickly

LP Graphical Interpretation

- The LP constraints define a polyhedron in the solution space

A polyhedron can be unbounded

- This is a polytope if the polyhedron is bounded and nonempty
- The basic, feasible solutions are vertices of this polyhedron
- With the linear cost function the solution will be at one of vertices

APPENDIX 3B: LINEAR PROGRAMMING (LP) 11

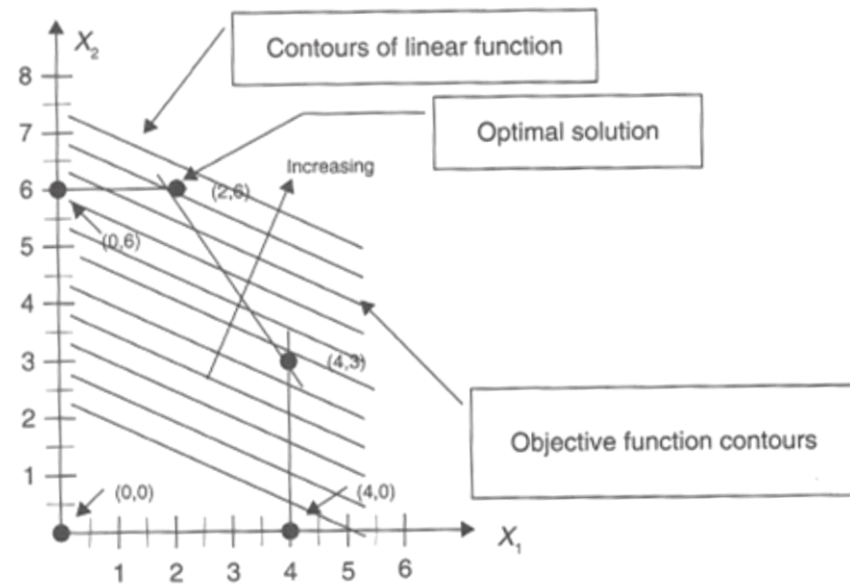


FIGURE 3.26 x_1, x_2 plane with cost contours and the optimal solution shown.

Simplex Algorithm



- The key is to move intelligently from one basic feasible solution (i.e., a vertex) to another, with the goal of continually decreasing the cost function
- The algorithm does this by determining the “best” variable to bring into the basis; this requires that another variable exit the basis, while always retaining a basic, feasible solution
- This is called pivoting

Determination of Variable to Enter the Basis



- To determine which non-basic variable should enter the basis (i.e., one which currently 0), look at how the cost function changes w.r.t. to a change in a non-basic variable (i.e., one that is currently zero)

$$\text{Define } \mathbf{z} = \mathbf{c}\mathbf{x} = [\mathbf{c}_B \quad \mathbf{c}_N] \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix}$$

$$\text{With } \mathbf{x}_B = \mathbf{A}_B^{-1} (\mathbf{b} - \mathbf{A}_N \mathbf{x}_N)$$

$$\text{Then } \mathbf{z} = \mathbf{c}_B \mathbf{A}_B^{-1} \mathbf{b} + (\mathbf{c}_N - \mathbf{c}_B \mathbf{A}_B^{-1} \mathbf{A}_N) \mathbf{x}_N$$

Elements of \mathbf{x}_n are all zero, but we are looking to change one to decrease the cost

Determination of Variable to Enter the Basis



- Define the reduced (or relative) cost coefficients as

$$\mathbf{r} = \mathbf{c}_N - \mathbf{c}_B \mathbf{A}_B^{-1} \mathbf{A}_N$$

\mathbf{r} is an n - m dimensional row vector

- Elements of this vector tell how the cost function will change for a change in a currently non-basic variable
- The variable to enter the basis is usually the one with the most negative relative cost
- If all the relative costs are nonnegative then we are at an optimal solution

Determination of Variable to Exit Basis



- The new variable entering the basis, say a position j , causes the values of all the other basic variables to change. In order to retain a basic, feasible solution, we need to insure no basic variables become negative. The change in the basic variables is given by

$$\mathbf{x}_B = \mathbf{x}_B - \mathbf{A}_B^{-1} \mathbf{a}_j \varepsilon$$

where ε is the value of the variable entering the basis, and \mathbf{a}_j is its associated column in \mathbf{A}

Determination of Variable to Exit Basis



We find the largest value ε such

$$\mathbf{x}_B = \mathbf{x}_B - \mathbf{A}_B^{-1} \mathbf{a}_j \varepsilon \geq \mathbf{0}$$

If no such ε exists then the problem is unbounded; otherwise at least one component of \mathbf{x}_B equals zero.

The associated variable exits the basis.

Canonical Form



- The Simplex Method works by having the problem in what is known as canonical form
- Canonical form is defined as having the m basic variables with the property that each appears in only one equation, its coefficient in that equation is unity, and none of the other basic variables appear in the same equation
- Sometime canonical form is readily apparent

$$\begin{array}{ll} \text{Minimize} & -(100x_1 + 120x_2) \\ \text{s.t.} & 2x_1 + 2x_2 + x_3 = 8 \\ & 3x_1 + 5x_2 + x_4 = 15 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

Note that with x_3 and x_4 as basic variables \mathbf{A}_B is the identity matrix

Canonical Form



- Other times canonical form is achieved by initially adding artificial variables to get an initial solution
- Example of the nutrition problem in canonical form with slack and artificial variables (denoted as \mathbf{y}) used to get an initial basic feasible solution

Let x_1 =ounces of A, x_2 = ounces of B

Minimize $y_1+y_2+y_3$

s.t. $2x_1 + 3x_2 + x_3 + y_1 = 20$

$$x_1 + 3x_2 - x_4 + y_2 = 12$$

$$4x_1 + 3x_2 - x_5 + y_3 = 24$$

$$x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3 \geq 0$$

Note that with $y_1, y_2,$
and y_3 as basic
variables \mathbf{A}_B is the
identity matrix

LP Tableau



- With the system in canonical form, the Simplex solution process can be illustrated by forming what is known as the LP tableau
 - Initially this corresponds to the \mathbf{A} matrix, with a column appended to include the \mathbf{b} vector, and a row added to give the relative cost coefficients; the last element is the negative of the cost function value
 - Define the tableau as \mathbf{Y} , with elements Y_{ij}
 - In canonical form the last column of the tableau gives the values of the basic variables
- During the solution the tableau is updated by pivoting

LP Tableau for the Nutrition Problem with Artificial Variables



- When in canonical form the relative costs vector is

$$\mathbf{r} = \mathbf{c}_N - \mathbf{c}_B \mathbf{A}_B^{-1} \mathbf{A}_N = \mathbf{c}_B \mathbf{A}_N$$

$$\mathbf{r} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T - [1 \quad 1 \quad 1] \begin{bmatrix} 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & -1 & 0 \\ 4 & 3 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -7 \\ -9 \\ -1 \\ 1 \\ 1 \end{bmatrix}^T$$

- The initial tableau for the artificial problem is then

x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
2	3	1	0	0	1	0	0	20
1	3	0	-1	0	0	1	0	12
4	3	0	0	-1	0	0	1	24
-7	-9	-1	1	1	0	0	0	-56

Note the last column gives the values of the basic variables

LP Tableau Pivoting



- Pivoting is used to move from one basic feasible solution to another
 - Select the pivot column (i.e., the variable coming into the basis, say q) as the one with the most negative relative cost
 - Select the pivot row (i.e., the variable going out of the basis) as the one with the smallest ratio of x_i/Y_{iq} for $Y_{iq} > 0$; define this as row p (x_i is given in the last column)

That is, we find the largest value ε such

$$\mathbf{x}_B = \mathbf{x}_B - \mathbf{A}_B^{-1} \mathbf{a}_q \varepsilon \geq \mathbf{0}$$

If no such ε exists then the problem is unbounded; otherwise at least one component of \mathbf{x}_B equals zero.

The associated variable exits the basis.

LP Tableau Pivoting for Nutrition Problem



- Starting at

x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
2	3	1	0	0	1	0	0	20
1	3	0	-1	0	0	1	0	12
4	3	0	0	-1	0	0	1	24
-7	-9	-1	1	1	0	0	0	-56

- Pivot on column $q=2$; for row get minimum of $\{20/3, 12/3, 24/3\}$, which is row $p=2$

LP Tableau Pivoting



- Pivoting on element Y_{pq} is done by
 - First dividing row p by Y_{pq} to change the pivot element to unity.
 - Then subtracting from the k^{th} row Y_{kq}/Y_{pq} times the p^{th} row for all rows with $Y_{kq} \neq 0$

x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
2	3	1	0	0	1	0	0	20
1	3	0	-1	0	0	1	0	12
4	3	0	0	-1	0	0	1	24
-7	-9	-1	1	1	0	0	0	-56

I'm only showing fractions with two ROD digits

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
	1	0	1	1	0	1	-1	0	8
Pivoting gives	0.33	1	0	-0.33	0	0	0.33	0	4
	3	0	0	1	-1	0	-1	1	12
	-4	0	-1	-2	1	0	3	0	-20

LP Tableau Pivoting, Example, cont.



- Next pivot on column 1, row 3

x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
1	0	1	1	0	1	-1	0	8
0.33	1	0	-0.33	0	0	0.33	0	4
3	0	0	1	-1	0	-1	1	12
-4	0	-1	-2	1	0	3	0	-20

- Which gives

x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
0	0	1	0.67	0.33	1	-0.67	-0.33	4
0	1	0	-0.44	0.11	0	0.44	-0.11	2.67
1	0	0	0.33	-0.33	0	-0.33	0.33	4.0
0	0	-1	-0.67	-0.33	0	1.67	1.33	-4

LP Tableau Pivoting, Example, cont.



- Next pivot on column 3, row 1

x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
0	0	1	0.67	0.33	1	-0.67	-0.33	4
0	1	0	-0.44	0.11	0	0.44	-0.11	2.67
1	0	0	0.33	-0.33	0	-0.33	0.33	4
0	0	-1	-0.67	-0.33	0	1.67	1.33	-4

- Which gives

x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	
0	0	1	0.67	0.33	1	-0.67	-0.33	4
0	1	0	-0.44	0.11	0	0.44	-0.11	2.67
1	0	0	0.33	-0.33	0	-0.33	0.33	4
0	0	0	0	0	1	1	1	0

Since there are no negative relative costs we are done with getting a starting solution

LP Tableau Full Problem



- The tableau from the end of the artificial problem is used as the starting point for the actual solution
 - Remove the artificial variables
 - Update the relative costs with the costs from the original problem and update the bottom right-hand corner value

$$\mathbf{c} = [0.2 \quad 0.25 \quad 0 \quad 0 \quad 0]$$

$$\mathbf{r} = \mathbf{c}_N - \mathbf{c}_B \mathbf{A}_B^{-1} \mathbf{A}_N = \mathbf{c}_B \mathbf{A}_N$$

$$\mathbf{r} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^T - [0 \quad 0.25 \quad 0.2] \begin{bmatrix} 0.67 & 0.33 \\ -0.44 & 0.11 \\ 0.33 & -0.33 \end{bmatrix} = \begin{bmatrix} 0.04 \\ 0.04 \end{bmatrix}^T$$

- Since none of the relative costs are negative we are done with $x_1=4$, $x_2=2.7$ and $x_3=4$