Homework #1, Problem #1

Thursday, September 9, 2021 9:38 AM

$$\dot{x}_1 = \frac{2}{3}x_1 - \frac{4}{3}x_1x_2$$

$$\dot{x}_2 = x_1x_2 - \frac{1}{2}x_2$$

<u>Initial</u> <u>Conditions</u>

$x_1(0) = 1 \quad x_2(0) = 1$

$\Delta t = 0.1$

Second Order Runge - Kutta

$$x(t+\Delta t) = x(t) + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = \Delta t\, f(x(t))$$

$$k_2 = \Delta t\, f(x(t) + k_1)$$

$$\dot{x}_1 = \frac{dx_1}{dt} = \frac{2}{3}x_1(t) - \frac{4}{3}x_1(t)\,x_2(t)$$

$$\dot{x}_2 = \frac{dx_2}{dt} = x_1(t)\,x_2(t) - \frac{1}{2}x_2(t)$$

$$\Rightarrow f(x(t)) = \begin{bmatrix} dx_1/dt \\ dx_2/dt \end{bmatrix} = \begin{bmatrix} \frac{2}{3}x_1(t) - \frac{4}{3}x_1(t)x_2(t) \\ x_1(t)x_2(t) - \frac{1}{2}x_2(t) \end{bmatrix}$$

Iteration #1

$$k_1 = \Delta t\, f(x(0)) = (0.1)\begin{bmatrix} \frac{2}{3}x_1(0) - \frac{4}{3}x_1(0)\,x_2(0) \\ x_1(0)\,x_2(0) - \frac{1}{2}x_2(0) \end{bmatrix}$$

$$= (0.1)\begin{bmatrix} \frac{2}{3}(1) - \frac{4}{3}(1)(1) \\ (1)(1) - \frac{1}{2}(1) \end{bmatrix} = (0.1)\begin{bmatrix} \frac{2}{3} - \frac{4}{3} \\ 1 - \frac{1}{2} \end{bmatrix}$$

$$= 0.1\begin{bmatrix} -2/3 \\ 1/2 \end{bmatrix} \approx \begin{bmatrix} -0.0667 \\ 0.05 \end{bmatrix}$$

$$k_2 = \Delta t\, f(x(0) + k_1) = \Delta t\, f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.0667 \\ 0.05 \end{bmatrix}\right) = \Delta t\, f\left(\begin{bmatrix} 0.933 \\ 1.05 \end{bmatrix}\right)$$

$$= 0.1 \begin{bmatrix} \frac{2}{3}(0.933) - \frac{4}{3}(0.933)(1.05) \\ (0.933)(1.05) - \frac{1}{2}(1.05) \end{bmatrix}$$

$$= 0.1 \begin{bmatrix} -0.684 \\ 0.455 \end{bmatrix} = \begin{bmatrix} -0.0684 \\ 0.0455 \end{bmatrix}$$

$$x(t+\Delta t) = x(t) + \frac{1}{2}(k_1 + k_2)$$

$$\Rightarrow x(0.1) = x(0) + \frac{1}{2}\left(\begin{bmatrix} -0.0667 \\ 0.05 \end{bmatrix} + \begin{bmatrix} -0.0684 \\ 0.0455 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\left(\begin{bmatrix} -0.135 \\ 0.0955 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.0676 \\ 0.0478 \end{bmatrix}$$

$$\Rightarrow x(0.1) = \begin{bmatrix} 0.9324 \\ 1.0478 \end{bmatrix}$$

This procedure is continued, and the results are summarized:

t	$x_1(t)$	$x_2(t)$
0	1	1
0.1	0.9324	1.0478
0.2	0.8642	1.0903
0.3	0.7968	1.1269
0.4	0.7313	1.1569

$$\Rightarrow \boxed{\begin{aligned} x_1(0.4) &= 0.7313 \\ x_2(0.4) &= 1.1569 \end{aligned}}$$

Homework #1, Problem #2

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$$\dot{x}_1 = \frac{2}{3}x_1 - \frac{4}{3}x_1x_2$$

$$\dot{x}_2 = x_1x_2 - \frac{1}{2}x_2$$

Initial Conditions

$x_1(0) = 1 \quad x_2(0) = 1$

$x_1(0.1) = 0.9324$

$x_2(0.1) = 1.0478$

$\Delta t = 0.1$

Second Order Adams - Bashforth

$$x(t+\Delta t) = x(t) + \frac{\Delta t}{2}\Big[3f(x(t)) - f(x(t-\Delta t))\Big] + O(\Delta t^3)$$

First Iteration $t = 0.1$, $\Delta t = 0.1$

$$\Rightarrow x(0.2) = x(0.1) + \frac{\Delta t}{2}\Big[3f(x(0.1)) - f(x(0.1-\Delta t))\Big]$$

$$= x(0.1) + \frac{\Delta t}{2}\Big[3f(x(0.1)) - f(x(0))\Big]$$

From problem #1 : $f(x(t)) = \begin{bmatrix} \frac{2}{3}x_1(t) - \frac{4}{3}x_1(t)x_2(t) \\ x_1(t)x_2(t) - \frac{1}{2}x_2(t) \end{bmatrix}$

$f(x(0)) = \begin{bmatrix} -0.667 \\ 0.5 \end{bmatrix} \qquad f(x(0.1)) = \begin{bmatrix} -0.681 \\ 0.453 \end{bmatrix}$

$$\Rightarrow x(0.2) = \begin{bmatrix} 0.9324 \\ 1.0478 \end{bmatrix} + \frac{0.1}{2}\left(3\begin{bmatrix} -0.681 \\ 0.453 \end{bmatrix} - \begin{bmatrix} -0.667 \\ 0.5 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0.9324 \\ 1.0478 \end{bmatrix} + 0.05\left(\begin{bmatrix} -2.043 \\ 1.359 \end{bmatrix} - \begin{bmatrix} -0.667 \\ 0.5 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0.9324 \\ 1.0478 \end{bmatrix} + 0.05\left(\begin{bmatrix} -1.376 \\ 0.859 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0.9324 \\ 1.0478 \end{bmatrix} + \begin{bmatrix} -0.0688 \\ 0.0429 \end{bmatrix} = \begin{bmatrix} 0.863 \\ 1.0907 \end{bmatrix}$$

This procedure is continued, and the results are summarized:

t	$x_1(t)$	$x_2(t)$
0	1	1
0.1	0.9324	1.0478
0.2	0.863	1.0907
0.3	0.796	1.128
0.4	0.729	1.158

$$\Rightarrow \boxed{\begin{aligned} X_1(0.4) &= 0.729 \\ X_2(0.4) &= 1.158 \end{aligned}}$$

Homework #1, Problem #3

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$$\dot{x}_1 = \frac{2}{3}x_1 - \frac{4}{3}x_1x_2 \qquad \dot{x}_2 = x_1x_2 - \frac{1}{2}x_2$$

Equilibrium points occur when $\dot{x}_1 = \dot{x}_2 = 0$

$$\frac{2}{3}x_1 - \frac{4}{3}x_1x_2 = 0 \Rightarrow x_1\left(\frac{2}{3} - \frac{4}{3}x_2\right) = 0$$

$$x_1x_2 - \frac{1}{2}x_2 = 0 \Rightarrow x_2\left(x_1 - \frac{1}{2}\right) = 0$$

$\Rightarrow$ Our system of equations is at equilibrium when:

$x_1 = 0$, $x_2 = 0$ OR $\frac{2}{3} - \frac{4}{3}x_2 = 0$, $x_1 - \frac{1}{2} = 0$

$$\Rightarrow \frac{2}{3} - \frac{4}{3}x_2 = 0 \Rightarrow \frac{4}{3}x_2 = \frac{2}{3} \Rightarrow x_2 = \frac{1}{2}$$

$$x_1 - 1/2 = 0 \Rightarrow x_1 = \frac{1}{2}$$

$\Rightarrow$ Two equilibrium points are: $\boxed{x_1 = 0,\ x_2 = 0}$ $\boxed{x_1 = 1/2,\ x_2 = 1/2}$

- Note that $x_1 = 0$, $x_2 = 0$ is a trivial solution. If you want a non-trivial solution, you can use Newton-Raphson, which gives you: $x_1 \approx -1.841 \times 10^{-14}$

$x_2 \approx -1.405 \times 10^{3}$

Homework #1, Problem #4

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$$\dot{x}_1 = 8(x_2 - x_1)$$

$$\dot{x}_2 = x_1(28 - x_3) - x_2$$

$$\dot{x}_3 = x_1 x_2 - \frac{4}{3} x_3$$

Initial Conditions

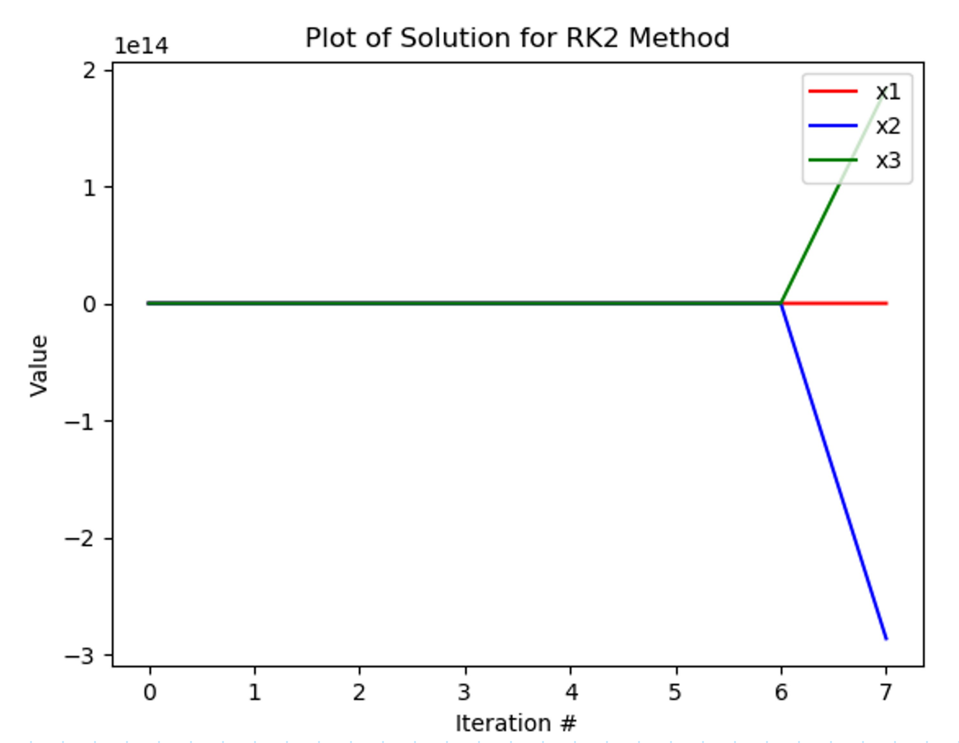
$x_1(0) = 5$

$x_2(0) = 5$ $\Delta t = 0.1$

$x_3(0) = 5$

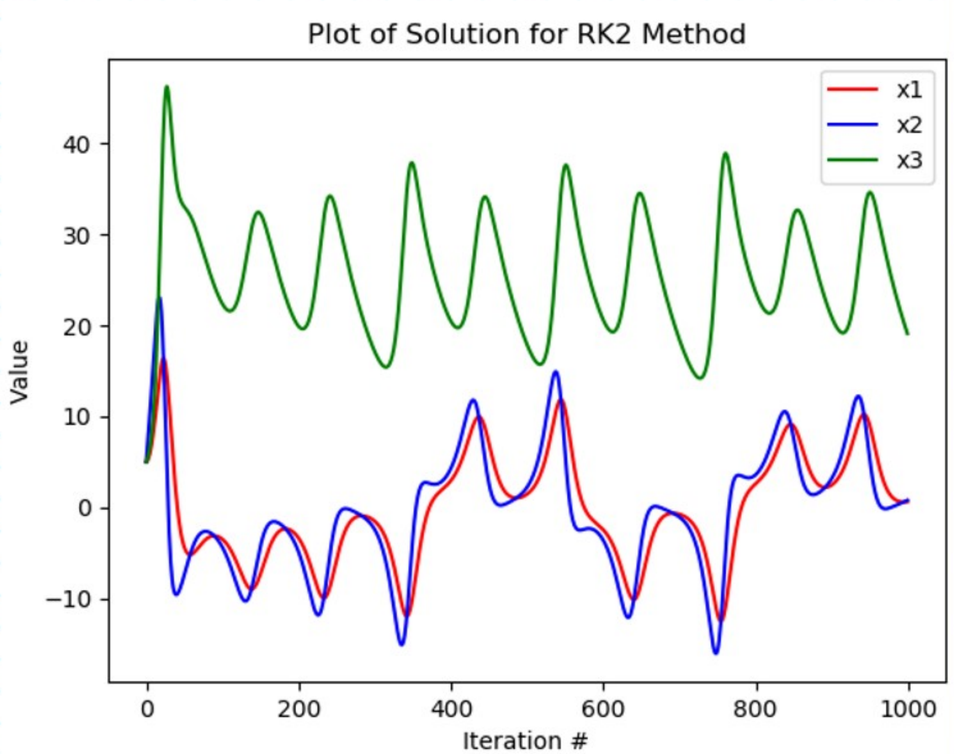
$$\Rightarrow f(x(t)) = \begin{bmatrix} 8(x_2(t) - x_1(t)) \\ x_1(t)(28 - x_3(t)) - x_2(t) \\ x_1(t)x_2(t) - \frac{4}{3}x_3(t) \end{bmatrix}$$

- Using $\Delta t = 0.1$, the estimates using RK2 exponentially increase, so that by 8 iterations ($t = 0.8$), the plot looks like:



Which indicates that we do not converge to an equilibrium point.

If we set $\Delta t = 0.01$, we get:



Thus, even at lower time steps, RK2 still does not converge.

Equilibrium points occur when $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0$

$$\Rightarrow \quad 8(x_2 - x_1) = 0 \qquad \Rightarrow \quad x_1 = x_2$$

$$x_1(28 - x_3) - x_2 = 0 \qquad x_1(28 - x_3) - x_1 = 0$$

$$x_1 x_2 - \frac{4}{3}x_3 = 0 \qquad x_1^2 - \frac{4}{3}x_3 = 0$$

$$\Rightarrow \quad x_1 = x_2$$

$$x_1\left[(28 - x_3) - 1\right] = 0 \quad \Rightarrow (28 - x_3) - 1 = 0 \quad \Rightarrow x_3 = 27$$

$$x_1^2 - \frac{4}{3}x_3 = 0 \quad \Rightarrow x_1^2 - \frac{4}{3}(27) \quad \Rightarrow \quad x_1^2 = 36 \Rightarrow x_1 = \pm 6$$

$\Rightarrow$ Possible equilibrium points:

$x_1 = 0$	$x_1 = 6$	$x_1 = -6$
$x_2 = 0$	$x_2 = 6$	$x_2 = -6$
$x_3 = 0$	$x_3 = 27$	$x_3 = 27$

Homework #1, Problem #5

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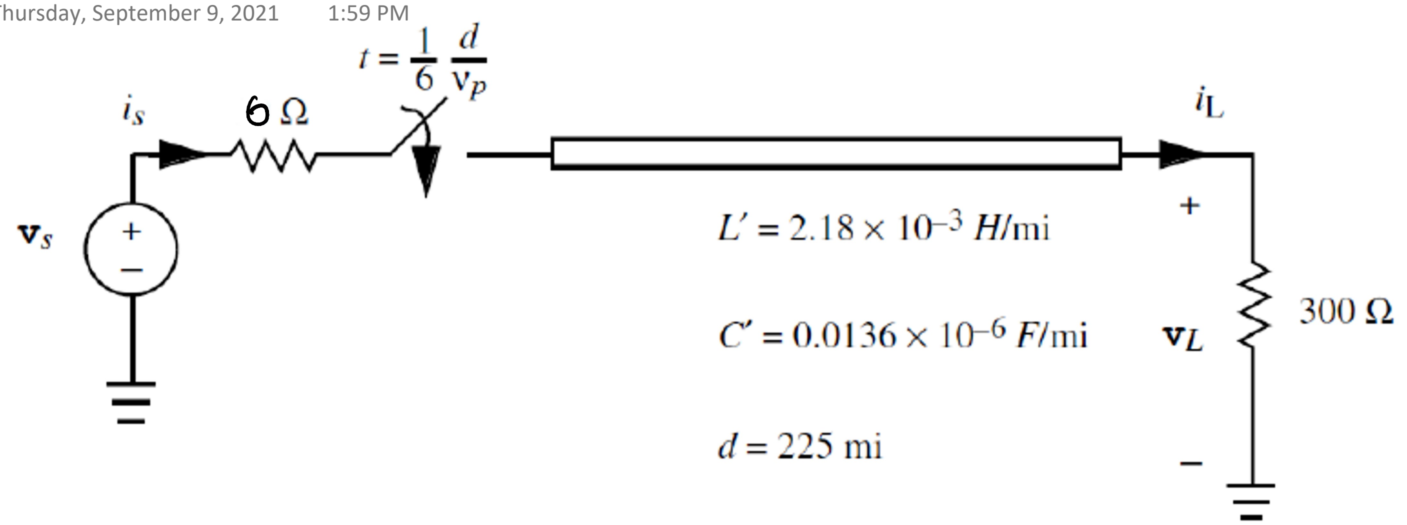


$\mathbf{v}_s = 188{,}000 \cos(2\pi 60\, t)$ volts

$0 \leq t \leq 0.4$

$\Delta t = \frac{1}{6}\frac{d}{v_p}$

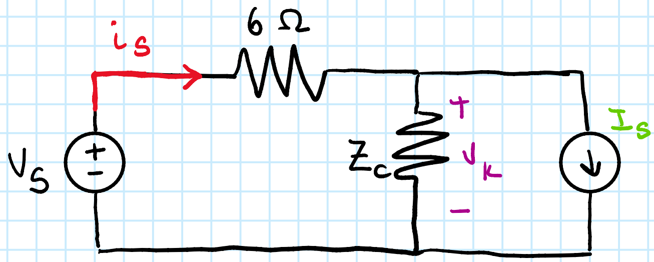
$Z_c' = \sqrt{L'/C'} \approx 400.367\,\Omega$

$v_p = \frac{1}{\sqrt{L'C'}} \approx 183{,}655$ mi/s

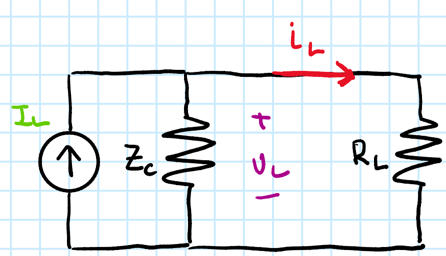
$\tau = d/v_p \approx 0.00123$ s

$\Delta t = 2.042 \times 10^{-4}$ s

Sending End Model



Receiving End Model



$I_s = i_L(t - d/v_p) - \frac{1}{Z_c} v_L(t - d/v_p)$

$I_L = i_s(t - d/v_p) + \frac{1}{Z_c} v_k(t - d/v_p)$

By circuit analysis:

$$v_K(t) = \frac{Z_c}{6+Z_c} V_s(t) \qquad i_L(t) = I_L(t) \frac{Z_c}{Z_c+R_L}$$

$$i_s(t) = \frac{V_s(t) - V_K(t)}{6} \qquad V_L(t) = Z_c \cdot \left(I_L(t) - i_L(t)\right)$$

We want to know: $v_L(t)$, $i_s(t)$, $i_L(t)$ over $0 \le t \le 4$

with $\Delta t = \frac{1}{6} \frac{d}{v_p}$

This is most easily achieved via code; the plots are found below: