ECEN 615
Methods of Electric Power Systems Analysis

Lecture 4: Power Flow

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Announcements

- Exam 1 is moved to Oct 13 (Oct 10 and 11 are fall break)
- Homework 1 is due on Thursday September 8
- ERCOT is having a Virtual College Day on September 16 at 930am (Central Time); details at ERCOT.COM/CAREERS
  - To register you can go to www.ercot.com/careers/edp

ERCOT Virtual College Day

To register you can go to www.ercot.com/careers/edp

An Interesting Video of an Uncleared Fault

- The below video shows an event from September 2 in the Netherlands (Flevoland providence) in which a transmission line fault was not cleared. Note the sag and smoke (steam?) coming from the line
- https://nos.nl/video/2443017-explosies-en-rokende-kabels-bij-verdeelstation-in-dronten

Quick PowerWorld Simulator Introduction

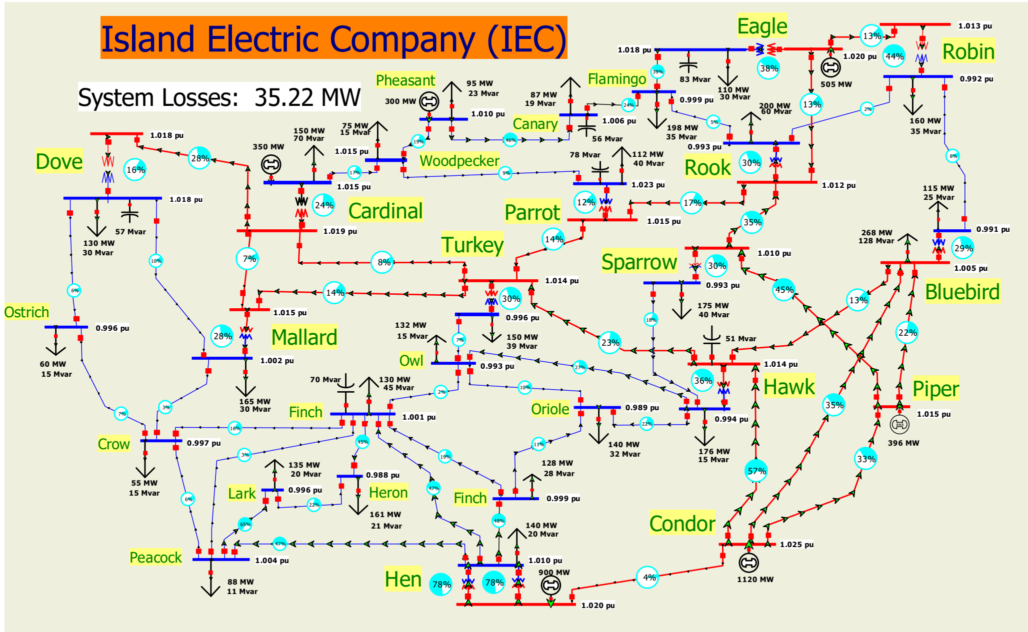
- Class will make extensive use of PowerWorld Simulator, initially using the 42 bus educational version
- Start getting familiar with PowerWorld, particularly the power flow basics; free training material is available at
  - www.powerworld.com/training/online-training
- By way of history, I started developing the power flow code that would become PowerWorld in 1987
- I also developed windows code for an power system dynamics short course in 1993
- PowerWorld was born in 1994 through the merging of these two code sets, originally to teach power system operators to utility industry non-technicals

Quick PowerWorld Simulator Introduction, cont.

- PowerWorld as a company was formed in 1996; the first commercial version was 4.0; current release version is 22 with Version 23 due out in fall
- The goal of PowerWorld Simulator (Simulator) is to make power system analysis as easy as possible; it started out mostly as a power flow, but has since gained lots of additional functionality, some of which we'll discuss in 615
  - Mostly it considers balanced, three-phase systems (there is some ability to handle unbalanced systems with fault analysis)

Quick PowerWorld Simulator Introduction, cont.

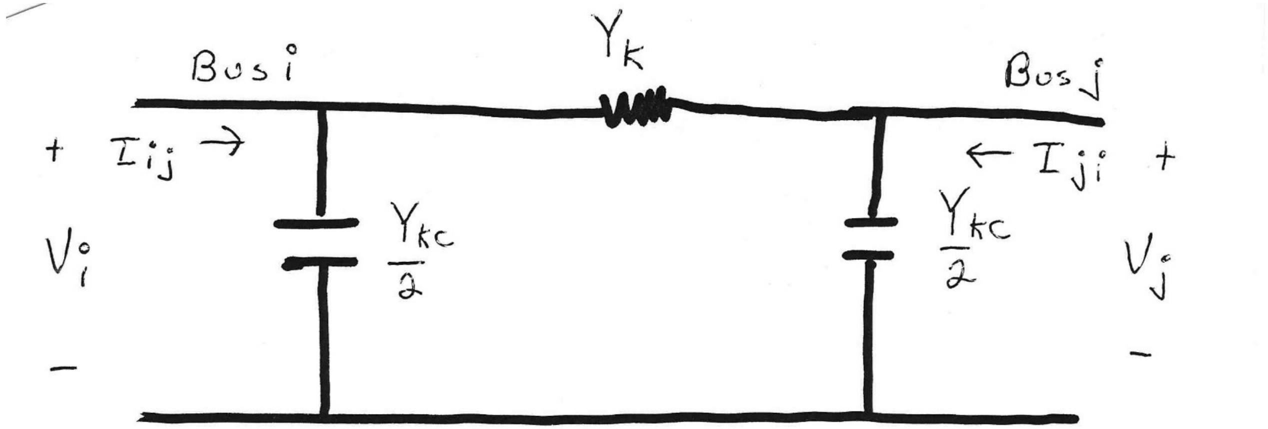
- There are two main types of files: *.pwb files define the power grid model, and *.pwd files contain the display files
- It has two main operating modes: Run Mode and Edit Mode; most 615 activities will use the Run Mode
- To solve the power flow for HW1 1) Open Simulator, 2) Select **File, Open Case**, 3) choose a *.pwb file, 4) Select **Tools** and click the green arrow to start the solution process, 5) click on the red (or green) circuit breakers to toggle the status, 6) right-click on a device to see details



Y_{bus} General Form

- The diagonal terms, Y_{ii}, are the self admittance terms, equal to the sum of the admittances of all devices incident to bus i.
- The off-diagonal terms, Y_{ij}, are equal to the negative of the sum of the admittances joining the two buses.
- With large systems Y_{bus} is a sparse matrix (that is, most entries are zero)
- Shunt terms, such as with the π line model, only affect the diagonal terms.

Modeling Shunts in the Y_{bus}

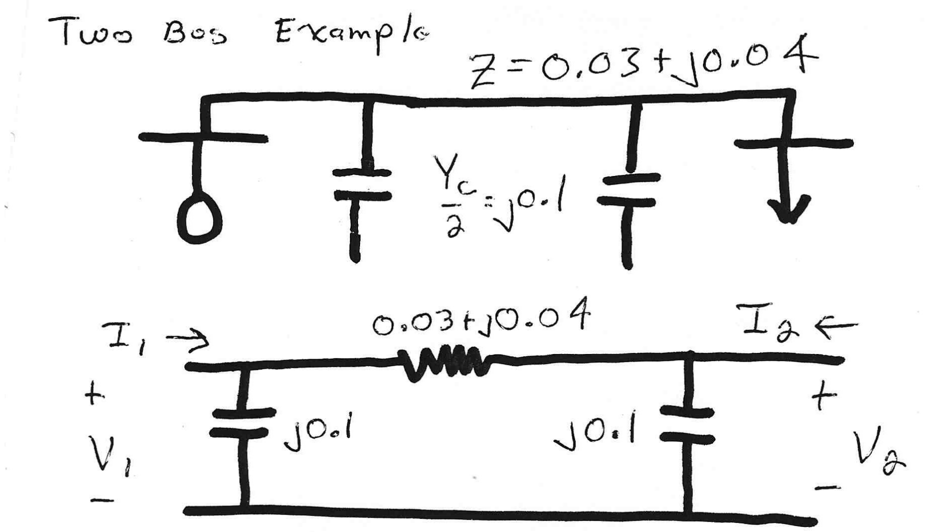


Since $I_{ij} = (V_i - V_j)Y_k + V_i \frac{Y_{kc}}{2}$

$$Y_{ii} = Y_{ii}^{\text{from other lines}} + Y_k + \frac{Y_{kc}}{2}$$

Note $Y_k = \frac{1}{Z_k} = \frac{1}{R_k + jX_k} \frac{R_k - jX_k}{R_k - jX_k} = \frac{R_k - jX_k}{R_k^2 + X_k^2}$

Two Bus System Example



$$I_1 = \frac{(V_1 - V_2)}{Z} + V_1 \frac{Y_c}{2} \qquad \frac{1}{0.03 + j0.04} = 12 - j16$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 12 - j15.9 & -12 + j16 \\ -12 + j16 & 12 - j15.9 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Using the Y_{bus}

If the voltages are known then we can solve for the current injections:

$$\mathbf{Y}_{bus}\mathbf{V} = \mathbf{I}$$

If the current injections are known then we can solve for the voltages:

$$\mathbf{Y}_{bus}^{-1}\mathbf{I} = \mathbf{V} = \mathbf{Z}_{bus}\mathbf{I}$$

where $\mathbf{Z}_{\text{bus}}$ is the bus impedance matrix

However, this requires that $\mathbf{Y}_{\text{bus}}$ not be singular; note it will be singular if there are no shunt connections!

Solving for Bus Currents

For example, in previous case assume

$$\mathbf{V} = \begin{bmatrix} 1.0 \\ 0.8 - j0.2 \end{bmatrix}$$

Then

$$\begin{bmatrix} 12 - j15.9 & -12 + j16 \\ -12 + j16 & 12 - j15.9 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.8 - j0.2 \end{bmatrix} = \begin{bmatrix} 5.60 - j0.70 \\ -5.58 + j0.88 \end{bmatrix}$$

Therefore the power injected at bus 1 is

$$\mathrm{S}_1 = V_1 I_1^* = 1.0 \times (5.60 + j0.70) = 5.60 + j0.70$$

$$S_2 = V_2 I_2^* = (0.8 - j0.2) \times (-5.58 - j0.88) = -4.64 + j0.41$$

Solving for Bus Voltages

For example, in previous case assume

$$\mathbf{I} = \begin{bmatrix} 5.0 \\ -4.8 \end{bmatrix}$$

Then

$$\begin{bmatrix} 12 - j15.9 & -12 + j16 \\ -12 + j16 & 12 - j15.9 \end{bmatrix}^{-1} \begin{bmatrix} 5.0 \\ -4.8 \end{bmatrix} = \begin{bmatrix} 0.0738 - j0.902 \\ -0.0738 - j1.098 \end{bmatrix}$$

Therefore the power injected is

$$\mathrm{S}_1 = V_1 I_1^* = (0.0738 - j0.902) \times 5 = 0.37 - j4.51$$

$$S_2 = V_2 I_2^* = (-0.0738 - j1.098) \times (-4.8) = 0.35 + j5.27$$

Power Flow Analysis

- When analyzing power systems we know neither the complex bus voltages nor the complex current injections
- Rather, we know the complex power being consumed by the load, and the power being injected by the generators plus their voltage magnitudes
- Therefore we can not directly use the Y_{bus} equations, but rather must use the power balance equations

Power Balance Equations

From KCL we know at each bus i in an n bus system the current injection, I_i, must be equal to the current that flows into the network

$$I_i = I_{Gi} - I_{Di} = \sum_{k=1}^{n} I_{ik}$$

Since $\mathbf{I} = \mathbf{Y}_{\text{bus}}\mathbf{V}$ we also know

$$I_i = I_{Gi} - I_{Di} = \sum_{k=1}^{n} Y_{ik} V_k$$

The network power injection is then $\mathrm{S}_\mathrm{i} = V_i I_i^*$

Power Balance Equations, cont'd

$$S_i = V_i I_i^* = V_i \left(\sum_{k=1}^{n} Y_{ik} V_k \right)^* = V_i \sum_{k=1}^{n} Y_{ik}^* V_k^*$$

This is an equation with complex numbers.

Sometimes we would like an equivalent set of real power equations. These can be derived by defining

$$Y_{ik} = G_{ik} + jB_{ik}$$

$$V_i = |V_i| e^{j\theta_i} = |V_i| \angle \theta_i$$

$$\theta_{ik} = \theta_i - \theta_k$$

Recall $e^{j\theta} = \cos\theta + j\sin\theta$

Real Power Balance Equations

$$S_i = P_i + jQ_i = V_i \sum_{k=1}^{n} Y_{ik}^* V_k^* = \sum_{k=1}^{n} |V_i||V_k| e^{j\theta_{ik}} (G_{ik} - jB_{ik})$$

$$= \sum_{k=1}^{n} |V_i||V_k| (\cos\theta_{ik} + j\sin\theta_{ik})(G_{ik} - jB_{ik})$$

Resolving into the real and imaginary parts

$$P_i = \sum_{k=1}^{n} |V_i||V_k|(G_{ik}\cos\theta_{ik} + B_{ik}\sin\theta_{ik}) = P_{Gi} - P_{Di}$$

$$Q_i = \sum_{k=1}^{n} |V_i||V_k|(G_{ik}\sin\theta_{ik} - B_{ik}\cos\theta_{ik}) = Q_{Gi} - Q_{Di}$$

Power Flow Analysis

- Classic paper for this lecture is W.F. Tinney and C.E. Hart, "Power Flow Solution by Newton's Method," *IEEE Power App System*, Nov 1967
- Basic power flow is also covered in essentially any power system analysis textbooks.
- We use the term "power flow" not "load flow" since power flows not load. Also, the power flow usage is not new (see title of Tinney's 1967 paper, and note Tinney references Ward's 1956 power flow paper)
  - A nice history of the power flow is given in an insert by Alvarado and Thomas in T.J. Overbye, J.D. Weber, "Visualizing the Electric Grid," *IEEE Spectrum*, Feb 2001.

Early Power Flow System Size

- In 1957 Bill Tinney, in a paper titled "Digital Solutions for Large Power Networks," studied a 100 bus, 200 branch system (with 2 KB of memory)!
- In Tinney's 1963 "Techniques for Exploiting Sparsity of the Network Admittance Matrix" paper (which gave us the Tinney Schemes 1, 2, and 3), uses 32 kB for 1000 nodes.
- In Tinney's classic 1967 "Power Flow Solution by Newton's Method" paper he applies his method to systems with up to about 1000 buses (with 32 kB of memory) and provides a solution time of 51 seconds for a 487 bus system.

Slack Bus

- We can not arbitrarily specify S at all buses because total generation must equal total load + total losses
- We also need an angle reference bus.
- To solve these problems we define one bus as the "slack" bus. This bus has a fixed voltage magnitude and angle, and a varying real/reactive power injection.
- In an actual power system the slack bus does not really exist; frequency changes locally when the power supplied does not match the power consumed

Three Types of Power Flow Buses

- There are three main types of power flow buses
  - Load (PQ) at which P/Q are fixed; iteration solves for voltage magnitude and angle.
  - Slack at which the voltage magnitude and angle are fixed; iteration solves for P/Q injections
  - Generator (PV) at which P and |V| are fixed; iteration solves for voltage angle and Q injection

Newton-Raphson Algorithm

- Most common technique for solving the power flow problem is to use the Newton-Raphson algorithm
- Key idea behind Newton-Raphson is to use sequential linearization

General form of problem: Find an $\mathbf{x}$ such that

$$\mathbf{f}(\hat{\mathbf{x}}) = 0$$

Newton-Raphson Method (scalar)

1. For each guess of $\hat{x}$, $x^{(v)}$, define

   $\Delta x^{(v)} = \hat{x} - x^{(v)}$

2. Represent $f(\hat{x})$ by a Taylor series about $f(x)$

$$f(\hat{x}) = f(x^{(v)}) + \frac{df(x^{(v)})}{dx}\Delta x^{(v)} + \frac{1}{2}\frac{d^2 f(x^{(v)})}{dx^2}\left(\Delta x^{(v)}\right)^2 + \text{higher order terms}$$

Newton-Raphson Method, cont'd

3. Approximate $f(\hat{x})$ by neglecting all terms except the first two

$$f(\hat{x}) = 0 \approx f(x^{(v)}) + \frac{df(x^{(v)})}{dx} \Delta x^{(v)}$$

4. Use this linear approximation to solve for $\Delta x^{(v)}$

$$\Delta x^{(v)} = -\left[\frac{df(x^{(v)})}{dx} \right]^{-1} f(x^{(v)})$$

5. Solve for a new estimate of x̂

$$x^{(v+1)} = x^{(v)} + \Delta x^{(v)}$$

Newton-Raphson Example

Use Newton-Raphson to solve $f(x) = x^2 - 2 = 0$

The equation we must iteratively solve is

$$\Delta x^{(v)} = -\left[\frac{df(x^{(v)})}{dx}\right]^{-1} f(x^{(v)})$$

$$\Delta x^{(v)} = -\left[\frac{1}{2x^{(v)}}\right]((x^{(v)})^2 - 2)$$

$$x^{(v+1)} = x^{(v)} + \Delta x^{(v)}$$

$$x^{(v+1)} = x^{(v)} - \left[\frac{1}{2x^{(v)}}\right]((x^{(v)})^2 - 2)$$

Newton-Raphson Example, cont'd

$$x^{(v+1)} = x^{(v)} - \left[\frac{1}{2x^{(v)}}\right]((x^{(v)})^2 - 2)$$

Guess $x^{(0)} = 1$. Iteratively solving we get

v	$x^{(v)}$	$f(x^{(v)})$	$\Delta x^{(v)}$
0	1	-1	0.5
1	1.5	0.25	-0.08333
2	1.41667	6.953×10^{-3}	-2.454×10^{-3}
3	1.41422	6.024×10^{-6}	

Newton-Raphson Power Flow

In the Newton-Raphson power flow we use Newton's method to determine the voltage magnitude and angle at each bus in the power system.

We need to solve the power balance equations

$$\mathrm{P_i} = \sum_{k=1}^{n} |V_i||V_k|(G_{ik}\cos\theta_{ik} + B_{ik}\sin\theta_{ik}) = P_{Gi} - P_{Di}$$

$$\mathrm{Q_i} = \sum_{k=1}^{n} |V_i||V_k|(G_{ik}\sin\theta_{ik} - B_{ik}\cos\theta_{ik}) = Q_{Gi} - Q_{Di}$$

Power Flow Variables

Assume the slack bus is the first bus (with a fixed voltage angle/magnitude).

We then need to determine the voltage angle/magnitude at the other buses.

$$\mathbf{x} = \begin{bmatrix} \theta_2 \\ \vdots \\ \theta_{\mathrm{n}} \\ |V_2| \\ \vdots \\ |V_n| \end{bmatrix} \qquad \mathbf{f}(\mathbf{x}) = \begin{bmatrix} P_2(\mathbf{x}) - P_{G2} + P_{D2} \\ \vdots \\ P_n(\mathbf{x}) - P_{Gn} + P_{Dn} \\ Q_2(\mathbf{x}) - Q_{G2} + Q_{D2} \\ \vdots \\ Q_n(\mathbf{x}) - Q_{Gn} + Q_{Dn} \end{bmatrix}$$

N-R Power Flow Solution

The power flow is solved using the same procedure discussed with the general Newton-Raphson:

Set $v = 0$; make an initial guess of $\mathbf{x}$, $\mathbf{x}^{(v)}$

While $\left\| \mathbf{f}(\mathbf{x}^{(v)}) \right\| > \varepsilon$ Do

$$\mathbf{x}^{(v+1)} = \mathbf{x}^{(v)} - \mathbf{J}(\mathbf{x}^{(v)})^{-1}\mathbf{f}(\mathbf{x}^{(v)})$$

$$v = v + 1$$

End While

Power Flow Jacobian Matrix

The most difficult part of the algorithm is determining and inverting the n by n Jacobian matrix, $\mathbf{J}(\mathbf{x})$

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \frac{\partial f_1(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\mathbf{x})}{\partial x_n} \\ \frac{\partial f_2(\mathbf{x})}{\partial x_1} & \frac{\partial f_2(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_2(\mathbf{x})}{\partial x_n} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial f_n(\mathbf{x})}{\partial x_1} & \frac{\partial f_n(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_n(\mathbf{x})}{\partial x_n} \end{bmatrix}$$

Power Flow Jacobian Matrix, cont'd

Jacobian elements are calculated by differentiating each function, $f_i(\mathbf{x})$, with respect to each variable. For example, if $f_i(\mathbf{x})$ is the bus i real power equation

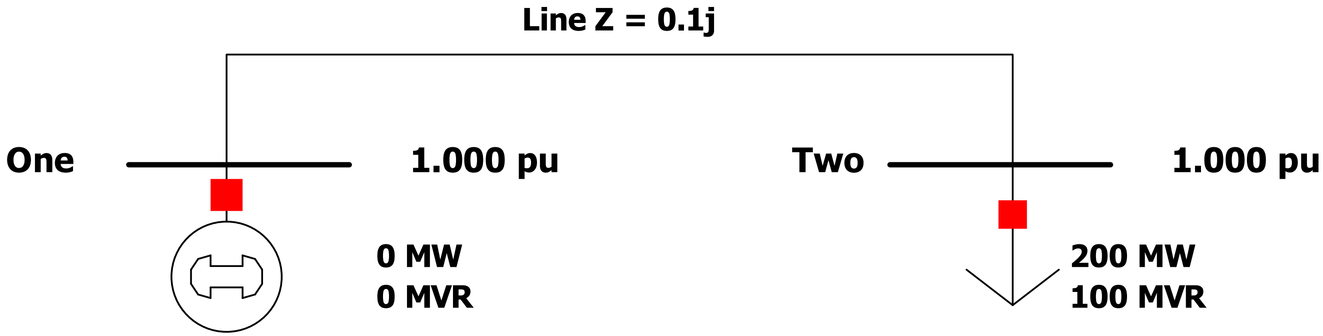
$$f_i(x) = \sum_{k=1}^{n} |V_i||V_k|(G_{ik}\cos\theta_{ik} + B_{ik}\sin\theta_{ik}) - P_{Gi} + P_{Di}$$

$$\frac{\partial f_i(x)}{\partial \theta_i} = \sum_{\substack{k=1 \\ k\neq i}}^{n} |V_i||V_k|(-G_{ik}\sin\theta_{ik} + B_i\cos\theta_{ik})$$

$$\frac{\partial f_i(x)}{\partial \theta_j} = |V_i||V_j|(G_{ij}\sin\theta_{ij} - B_{ij}\cos\theta_{ij}) \ \ (j \neq i)$$

Two Bus Newton-Raphson Example

- For the two bus power system shown below, use the Newton-Raphson power flow to determine the voltage magnitude and angle at bus two. Assume that bus one is the slack and S_{Base} = 100 MVA.



$$\mathbf{x} = \begin{bmatrix} \theta_2 \\ |V_2| \end{bmatrix} \qquad \mathbf{Y}_{bus} = \begin{bmatrix} -j10 & j10 \\ j10 & -j10 \end{bmatrix}$$

Two Bus Example, cont'd

General power balance equations

$$P_i = \sum_{k=1}^{n} |V_i||V_k|(G_{ik}\cos\theta_{ik} + B_{ik}\sin\theta_{ik}) = P_{Gi} - P_{Di}$$

$$Q_i = \sum_{k=1}^{n} |V_i||V_k|(G_{ik}\sin\theta_{ik} - B_{ik}\cos\theta_{ik}) = Q_{Gi} - Q_{Di}$$

Bus two power balance equations

$$|V_2||V_1|(10\sin\theta_2) + 2.0 = 0$$

$$|V_2||V_1|(-10\cos\theta_2) + |V_2|^2(10) + 1.0 = 0$$

Two Bus Example, cont'd

$$\mathrm{P}_2(\mathbf{x}) = |V_2|(10\sin\theta_2) + 2.0 = 0$$

$$Q_2(\mathbf{x}) = |V_2|(-10\cos\theta_2) + |V_2|^2(10) + 1.0 = 0$$

Now calculate the power flow Jacobian

$$J(\mathbf{x}) = \begin{bmatrix} \dfrac{\partial \mathrm{P}_2(\mathbf{x})}{\partial \theta_2} & \dfrac{\partial \mathrm{P}_2(\mathbf{x})}{\partial |V|_2} \\ \dfrac{\partial \mathrm{Q}_2(\mathbf{x})}{\partial \theta_2} & \dfrac{\partial \mathrm{Q}_2(\mathbf{x})}{\partial |V|_2} \end{bmatrix}$$

$$= \begin{bmatrix} 10|V_2|\cos\theta_2 & 10\sin\theta_2 \\ 10|V_2|\sin\theta_2 & -10\cos\theta_2 + 20|V_2| \end{bmatrix}$$

Two Bus Example, First Iteration

Set $v = 0$, guess $\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Calculate

$$f(\mathbf{x}^{(0)}) = \begin{bmatrix} |V_2|(10\sin\theta_2) + 2.0 \\ |V_2|(-10\cos\theta_2) + |V_2|^2(10) + 1.0 \end{bmatrix} = \begin{bmatrix} 2.0 \\ 1.0 \end{bmatrix}$$

$$\mathbf{J}(\mathbf{x}^{(0)}) = \begin{bmatrix} 10|V_2|\cos\theta_2 & 10\sin\theta_2 \\ 10|V_2|\sin\theta_2 & -10\cos\theta_2 + 20|V_2| \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

Solve $\mathbf{x}^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 2.0 \\ 1.0 \end{bmatrix} = \begin{bmatrix} -0.2 \\ 0.9 \end{bmatrix}$

Two Bus Example, Next Iterations

$$f(\mathbf{x}^{(1)}) = \begin{bmatrix} 0.9(10\sin(-0.2)) + 2.0 \\ 0.9(-10\cos(-0.2)) + 0.9^2 \times 10 + 1.0 \end{bmatrix} = \begin{bmatrix} 0.212 \\ 0.279 \end{bmatrix}$$

$$\mathbf{J}(\mathbf{x}^{(1)}) = \begin{bmatrix} 8.82 & -1.986 \\ -1.788 & 8.199 \end{bmatrix}$$

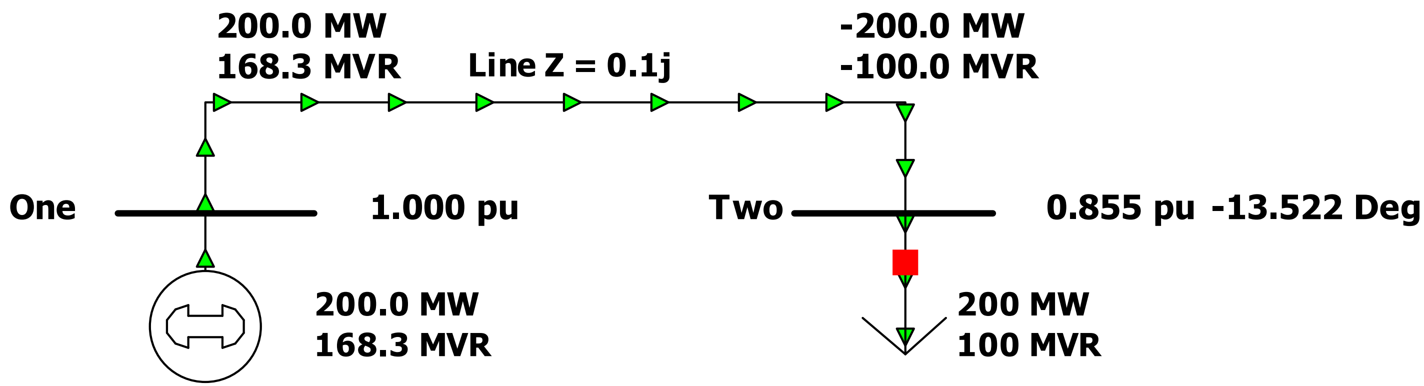
$$\mathbf{x}^{(2)} = \begin{bmatrix} -0.2 \\ 0.9 \end{bmatrix} - \begin{bmatrix} 8.82 & -1.986 \\ -1.788 & 8.199 \end{bmatrix}^{-1} \begin{bmatrix} 0.212 \\ 0.279 \end{bmatrix} = \begin{bmatrix} -0.233 \\ 0.8586 \end{bmatrix}$$

$$f(\mathbf{x}^{(2)}) = \begin{bmatrix} 0.0145 \\ 0.0190 \end{bmatrix} \qquad \mathbf{x}^{(3)} = \begin{bmatrix} -0.236 \\ 0.8554 \end{bmatrix}$$

$$f(\mathbf{x}^{(3)}) = \begin{bmatrix} 0.0000906 \\ 0.0001175 \end{bmatrix} \qquad \text{Done!} \qquad V_2 = 0.8554\angle -13.52°$$

Two Bus Solved Values

- Once the voltage angle and magnitude at bus 2 are known we can calculate all the other system values, such as the line flows and the generator reactive power



PowerWorld Case Name: Bus2_Intro
Note, most PowerWorld cases will be available on the course website

Two Bus Case Low Voltage Solution

This case actually has two solutions! The second "low voltage" solution is found by using a low initial guess.

Set $v = 0$, guess $\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0.25 \end{bmatrix}$

Calculate

$$f(\mathbf{x}^{(0)}) = \begin{bmatrix} |V_2|(10\sin\theta_2) + 2.0 \\ |V_2|(-10\cos\theta_2) + |V_2|^2(10) + 1.0 \end{bmatrix} = \begin{bmatrix} 2 \\ -0.875 \end{bmatrix}$$

$$\mathbf{J}(\mathbf{x}^{(0)}) = \begin{bmatrix} 10|V_2|\cos\theta_2 & 10\sin\theta_2 \\ 10|V_2|\sin\theta_2 & -10\cos\theta_2 + 20|V_2| \end{bmatrix} = \begin{bmatrix} 2.5 & 0 \\ 0 & -5 \end{bmatrix}$$

Low Voltage Solution, cont'd

$$\text{Solve } \mathbf{x}^{(1)} = \begin{bmatrix} 0 \\ 0.25 \end{bmatrix} - \begin{bmatrix} 2.5 & 0 \\ 0 & -5 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -0.875 \end{bmatrix} = \begin{bmatrix} -0.8 \\ 0.075 \end{bmatrix}$$

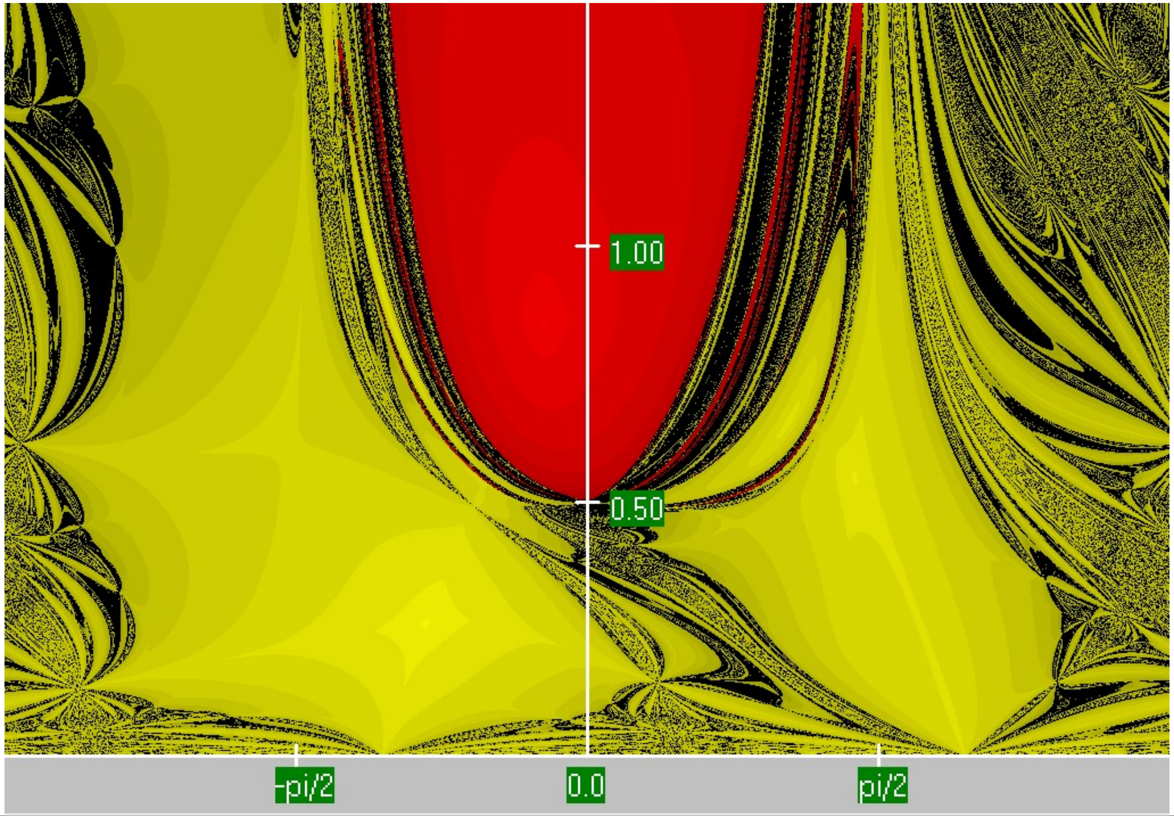
$$\mathbf{f}(\mathbf{x}^{(2)}) = \begin{bmatrix} 1.462 \\ 0.534 \end{bmatrix} \quad \mathbf{x}^{(2)} = \begin{bmatrix} -1.42 \\ 0.2336 \end{bmatrix} \qquad \mathbf{x}^{(3)} = \begin{bmatrix} -0.921 \\ 0.220 \end{bmatrix}$$

Practical Power Flow Software Note

- Most commercial software packages have built in defaults to prevent convergence to low voltage solutions.
  - One approach is to automatically change the load model from constant power to constant current or constant impedance when the load bus voltage gets too low
  - In PowerWorld these defaults can be modified on the Tools, Simulator Options, Advanced Options page; note you also need to disable the "Initialize from Flat Start Values" option
  - The PowerWorld case Bus2_Intro_Low is set solved to the low voltage solution
  - Initial bus voltages can be set using the Bus Information Dialog

Two Bus Region of Convergence

- Slide shows the region of convergence for different initial guesses of bus 2 angle (x-axis) and magnitude (y-axis)



The red region shows the initial guesses that converge to the high voltage solution, while the yellow region shows the guesses that converge to the low voltage solution

Power Flow Fractal Region of Convergence

- Earliest paper showing fractal power flow regions of convergence is by C.L DeMarco and T.J. Overbye, “Low Voltage Power Flow Solutions and Their Role in Exit Time Bases Security Measures for Voltage Collapse,” *Proc. 27th IEEE CDC*, December 1988
- A more widely known paper is J.S. Thorp, S.A. Naqavi, “Load-Flow Fractals Draw Clues to Erratic Behavior,” *IEEE Computer Applications in Power*, January 1997

PV Buses

- Since the voltage magnitude at PV buses is fixed there is no need to explicitly include these voltages in **x** or write the reactive power balance equations
  - the reactive power output of the generator varies to maintain the fixed terminal voltage (within limits)
  - optionally these variations/equations can be included by just writing the explicit voltage constraint for the generator bus

$$|V_i| - V_{i\ setpoint} = 0$$

Three Bus PV Case Example

For this three bus case we have

$$\mathbf{x} = \begin{bmatrix} \theta_2 \\ \theta_3 \\ |V_2| \end{bmatrix} \qquad \mathbf{f}(\mathbf{x}) = \begin{bmatrix} P_2(\mathbf{x}) - P_{G2} + P_{D2} \\ P_3(\mathbf{x}) - P_{G3} + P_{D3} \\ Q_2(\mathbf{x}) + Q_{D2} \end{bmatrix} = 0$$