ECEN 615
Methods of Electric Power Systems Analysis

Lecture 18: QR, Equivalents

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Announcements

- Skim Chapters 3, 4 and 5
- Starting reading Chapter 8
- Homework 6 is due Thursday Nov 10

QR Factorization

- Used in SE since it handles ill-conditioned m by n matrices (with m >= n)
- Can be used with sparse matrices
- We will first split the $\mathbf{R}^{-1}$ matrix

$$\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H} = \mathbf{H}^T\mathbf{R}^{-1/2}\mathbf{R}^{-1/2}\mathbf{H} = \mathbf{H}'^T\mathbf{H}'$$

- QR factorization represents the m by n **H'** matrix as

$$\mathbf{H}' = \mathbf{Q}\,\mathbf{U}$$

with **Q** an m by m orthonormal matrix and **U** an upper triangular matrix (most books use **Q R** but we use **U** to avoid confusion with the previous **R**)

Orthonormal Matrices

- The term orthogonal is used with vectors to indicate their dot product is zero (i.e., they are perpendicular to each other)
- Orthonormal is used to indicate they are orthogonal and each has unit length (magnitude of 1)
- The definition of an orthogonal matrix is $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$
  - This implies its inverse always exists
- Its determinant is 1
- They can be used for transformations such as an angular rotation

QR Factorization

- If $\mathbf{H}' = \mathbf{Q}\,\mathbf{U}$ then $\mathbf{H}'^T\mathbf{H}' = \mathbf{U}^T\mathbf{Q}^T\mathbf{Q}\,\mathbf{U}$
- But since $\mathbf{Q}$ is an orthonormal matrix, $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$
- Hence we have $\mathbf{H}'^T\mathbf{H}' = \mathbf{U}^T\,\mathbf{U}$

Originally $\Delta\mathbf{x} = \left[\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right]^{-1}\mathbf{H}^T\mathbf{R}^{-1}\left[\mathbf{z}^{meas} - \mathbf{f}(\mathbf{x})\right]$

With $\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H} = \mathbf{H}'^T\mathbf{H}' = \mathbf{H}'^T\mathbf{H}' = \mathbf{U}^T\mathbf{U}$

Let $\hat{\mathbf{z}} = \mathbf{Q}^T\mathbf{R}^{-1/2}\left[\mathbf{z}^{meas} - \mathbf{f}(\mathbf{x})\right]$

$$\Delta\mathbf{x} = \left[\mathbf{U}^T\mathbf{U}\right]^{-1}\mathbf{H}^T\mathbf{R}^{-1/2}\mathbf{R}^{-1/2}\left[\mathbf{z}^{meas} - \mathbf{f}(\mathbf{x})\right] = \left[\mathbf{U}^T\mathbf{U}\right]^{-1}\mathbf{U}^T\hat{\mathbf{z}}$$

$$\mathbf{U}^T\mathbf{U}\Delta\mathbf{x} = \mathbf{U}^T\hat{\mathbf{z}} \rightarrow \Delta\mathbf{x} = \mathbf{U}^{-1}\hat{\mathbf{z}}$$

Q is an m by m matrix

QR Factorization

- When factored the **U** matrix (i.e., what most call the **R** matrix) will be an m by n upper triangular matrix
- Several methods are available including the Householder method and the Givens method
- Givens is preferred when dealing with sparse matrices
- A good reference is Gene H. Golub and Charles F. Van Loan, “Matrix Computations,” third edition, Johns Hopkins University Press, 1996

Givens Algorithm for Factoring a Matrix A

- The Givens algorithm works by pre-multiplying the initial matrix, **A**, by a series of matrices and their transposes, starting with $\mathbf{G}_1\mathbf{G}_1^{\mathrm{T}}$
  - If **A** is m by n, then each **G** is an m by m matrix
- The algorithm proceeds column by column, sequentially zeroing out elements in the lower triangle of **A**, starting at the bottom of each column

$$\mathbf{G}_1 \ldots \mathbf{G}_p \mathbf{G}_p^T \ldots \mathbf{G}_1^T \mathbf{A} = \mathbf{Q}\mathbf{U}$$

$$\mathbf{G}_1 \ldots \mathbf{G}_p = \mathbf{Q}$$

$$\mathbf{G}_p^T \ldots \mathbf{G}_1^T \mathbf{A} = \mathbf{U}$$

If **A** is sparse, then we can take advantage of sparsity going up the column

Givens Algorithm

- To zero out element **A**[i,j], with i > j we first solve with a=A[k,j], b= A[i,j]

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix} \qquad r = \sqrt{a^2 + b^2}$$

To zero out an element we need a non-zero pivot element in column j; assume this row as k.

- A numerically safe algorithm is

If b=0 then c=1, s=0 // i.e, no rotation is needed

Else If |b|>|a| then $\tau = -a/b; \quad s = 1/\sqrt{1+\tau^2}; c = s\tau$

Else $\tau = -b/a; \quad c = 1/\sqrt{1+\tau^2}; s = c\tau$

Givens G Matrix

- The orthogonal **G**(i,k,θ) matrix is then

$$\mathbf{G}(i,\mathrm{k},\theta)=\begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & -s & \cdots & c & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

As noted, to zero out an element we need a non-zero pivot element in column j; assume this row as k. Row k here is the first non-zero above row i.

- Premultiplication by $\mathbf{G}(i,k,\theta)^T$ is a rotation by θ radians in the (i,k) coordinate plane

Small Givens Example

- Let

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 1 & 0 \\ 0 & 5 \\ 2 & 1 \end{bmatrix}$$

First start in column j=1; we will zero out A[4,1] with i=4, k=2

- First we zero out **A**[4,1], a=1, b=2 giving s= 0.8944, c=-0.4472

$$\mathbf{G}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.4472 & 0 & 0.8944 \\ 0 & 0 & 1 & 0 \\ 0 & -0.8944 & 0 & -0.4472 \end{bmatrix} \quad \mathbf{G}_1^T\mathbf{A} = \begin{bmatrix} 4 & 2 \\ -2.236 & -0.8944 \\ 0 & 5 \\ 0 & -0.4472 \end{bmatrix}$$

Small Givens Example

- Next zero out A[2,1] with a=4, b=-2.236, giving c= -0.8729, s=0.4880

$$\mathbf{G}_2 = \begin{bmatrix} 0.873 & 0.488 & 0 & 0 \\ -0.488 & 0.873 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{G}_2^T\mathbf{G}_1^T\mathbf{A} = \begin{bmatrix} 4.58 & 2.18 \\ 0 & 0.195 \\ 0 & 5 \\ 0 & -0.447 \end{bmatrix}$$

k=1 with A[k,j]=4

- Next zero out A[4,2] with a=5, b=-0.447, c=0.996, s=0.089

$$\mathbf{G}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.996 & 0.089 \\ 0 & 0 & -0.089 & 0.996 \end{bmatrix} \quad \mathbf{G}_3^T\mathbf{G}_2^T\mathbf{G}_1^T\mathbf{A} = \begin{bmatrix} 4.58 & 2.18 \\ 0 & 0.195 \\ 0 & 5.020 \\ 0 & 0 \end{bmatrix}$$

j=2, k=3 with A[k,j]=5

Small Givens Example

- Next zero out A[3,2] with a=0.195, b=5.02, c=-0.039, s=0.999

$$\mathbf{G}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.039 & 0.999 & 0 \\ 0 & -0.999 & -0.039 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{G}_4^T\mathbf{G}_3^T\mathbf{G}_2^T\mathbf{G}_1^T\mathbf{A} = \mathbf{U} = \begin{bmatrix} 4.58 & 2.18 \\ 0 & -5.023 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

j=2, k=2 with
A[k,j]=0.195

- Also we have

$$\mathbf{Q} = \mathbf{G}_1\mathbf{G}_2\mathbf{G}_3\mathbf{G}_4 = \begin{bmatrix} 0.872 & -0.019 & 0.487 & 0 \\ 0.218 & 0.094 & -0.387 & 0.891 \\ 0 & -0.995 & -0.039 & 0.089 \\ 0.436 & -0.009 & -0.782 & -0.445 \end{bmatrix}$$

Start of Givens for SE Example

- Starting with the **H** matrix we get

$$\mathbf{H}' = \mathbf{R}^{-1/2}\mathbf{H} = 100 \times \begin{bmatrix} 0 & -10 & 0 \\ 10 & 0 & -10 \\ 0 & 10 & 0 \\ -10 & 0 & 10 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- To zero out **H'**[5,1]=1 we have b=100, a=-1000, giving c=0.995, s=0.0995

$$\mathbf{G}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.995 & 0.0995 & 0 \\ 0 & 0 & 0 & -0.0995 & 0.995 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Here the column (j) is 1, while i is 5 and k is 4.

Start of Givens for SE Example

- Which gives

$$\mathbf{G}_1^T \mathbf{H}' = 100 \times \begin{bmatrix} 0 & -10 & 0 \\ 10 & 0 & -10 \\ 0 & 10 & 0 \\ 10.049 & 0 & -9.95 \\ 0 & 0 & 0.995 \\ 0 & 0 & 1 \end{bmatrix}$$

- The next rotation would be to zero out element [4,1], continuing until all the elements in the lower triangle have been reduced

Givens Comments

- For a full matrix, Givens is $O(mn^2)$ since each element in the lower triangle needs to be zeroed $O(nm)$, and each operation is $O(n)$
- Computation can be drastically reduced for a sparse matrix since we only need to zero out the elements that are initially non-zero, and any that become non-zero (i.e., the fills)
  - Also, for each multiply we only need to deal with the nonzeros in the impacted row
- Givens rotation is commonly used to solve the SE

Power System Equivalents

- No electric grid model is ever going to completely represent a real electric grid
  - "All models are wrong but some models are useful"
- A key modeling consideration is how much of the electric grid to represent
  - For large-scale systems the distribution system is usually equivalenced at some point; this has few system level ramifications if it is radial; if it is networked then there are potential issues
  - At the transmission level either the full interconnect is represented or it is equivalenced
  - In an SE model in large grids (like the Eastern Interconnect) it is always an electrical equivalent

Kron Reduction, Ward Equivalents

- For decades power system network models have been equivalenced using the approach originally presented by J.B. Ward in 1949 AIEE paper “Equivalent Circuits for Power-Flow Studies”
  - Paper’s single reference is to 1939 book by Gabriel Kron, so this is also known as Kron’s reduction or a Ward equivalent
- System buses are partitioned into a study system (s) to be retained and an external system (e) to be eliminated; buses in study system that connect to the external are known as boundary buses

Ward Equivalents

- The Ward approach is based on the below relationship

$$\begin{bmatrix} I_s \\ I_e \end{bmatrix} = \begin{bmatrix} Y_{ss} & Y_{se} \\ Y_{es} & Y_{ee} \end{bmatrix} \begin{bmatrix} V_s \\ V_e \end{bmatrix}$$

$$\left(I_s - Y_{se} Y_{ee}^{-1} I_e \right) = \left(Y_{ss} - Y_{se} Y_{ee}^{-1} Y_{es} \right) V_s$$

- No impact on study, non-boundary buses
- Equivalent is created by doing a partial factorization of the $\mathbf{Y}_{\text{bus}}$
  - Computationally efficient

Other Types of Equivalents

- There are many different methods available for creating power system equivalents
  - A classic paper is by S. Deckmann, et. al., "Studies on Power System Load Flow Equivalencing," *IEEE Transactions Power App. And Syst*., Nov/Dec 1980
  - Companion paper covers numerical testing of equivalents
- The major equivalencing types are
  - Ward-Type Equivalence: this is what we'll be covering, with the major differences associated with how the generator buses and equivalent loads are represented
  - REI Equivalents: All boundary buses connect to one "REI" bus
  - Linearized Methods: Linearize about an operating point
  - Others: PTDF-based, backbone type

Equivalent System Properties

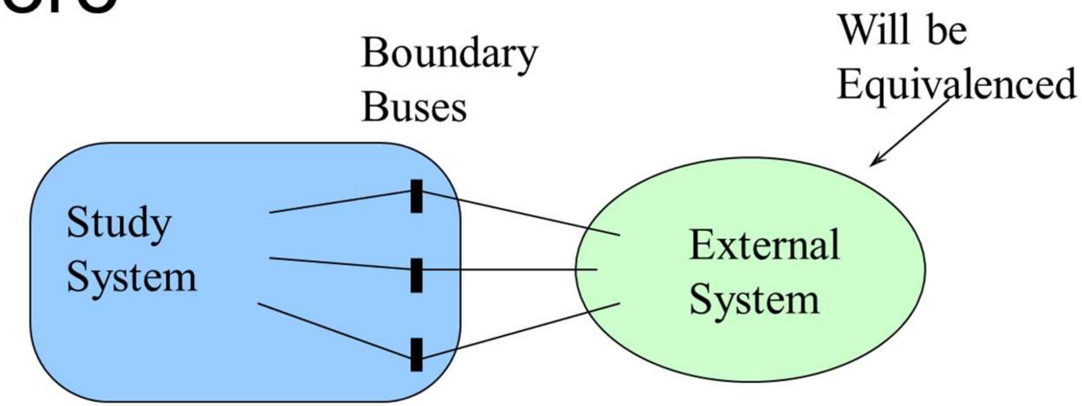
- An equivalent is usually created from a larger model
  - In the Eastern Interconnect there are full grid models that are used for wide-area planning, these are equivalenced for real-time usage or more specialized studies
- The equivalent is usually smaller and less detailed
  - Solves quicker
  - Requires less storage
  - Requires less up-to-date data
- Equivalences contain fictitious elements
  - This can make modeling/updating more difficult
- The equivalent only approximates the behavior of the original

Study vs External System

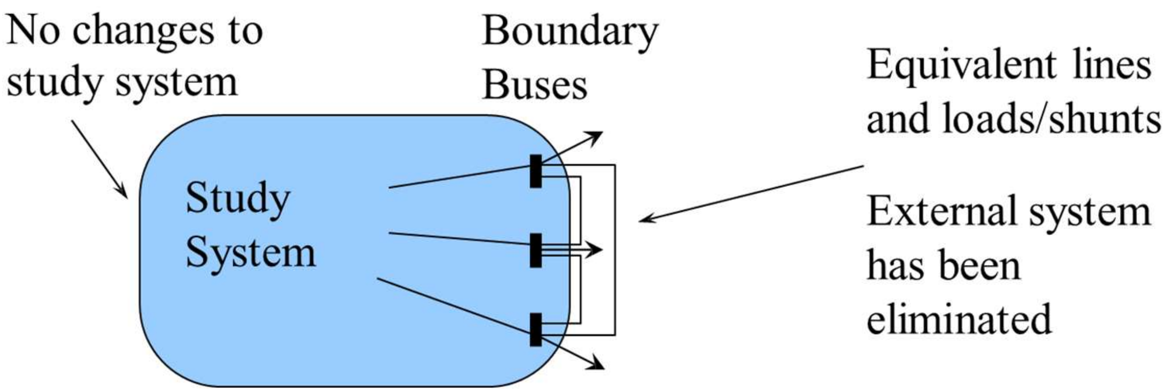
- The key decision in creating an equivalent is to divide the system into a study portion that is represented in detail, and an external portion that is represented by the equivalent
- The two systems are joined at boundary buses, which are part of the study subsystem
- How this is done is application specific; for example:
  - for real-time use it does not make sense to retain significant portions of the grid for which there is no real-time information
  - for contingency analysis the impact of the contingency is localized
  - for planning the new system additions have localized impacts

Ward Type Equivalencing

Before



After



Ward Type Equivalencing Considerations

- The Ward equivalent is calculated by doing a partial factorization of the $\mathbf{Y}_{\text{bus}}$
  - The equivalent buses are numbered before the study buses
  - As the equivalent buses are eliminated their first neighbors are joined together
  - At the end, many of the boundary buses are connected
  - This can GREATLY decrease the sparsity of the system
  - Buses with different voltages can be directly connected

$$\begin{bmatrix} I_s \\ I_e \end{bmatrix} = \begin{bmatrix} Y_{ss} & Y_{se} \\ Y_{es} & Y_{ee} \end{bmatrix} \begin{bmatrix} V_s \\ V_e \end{bmatrix}$$

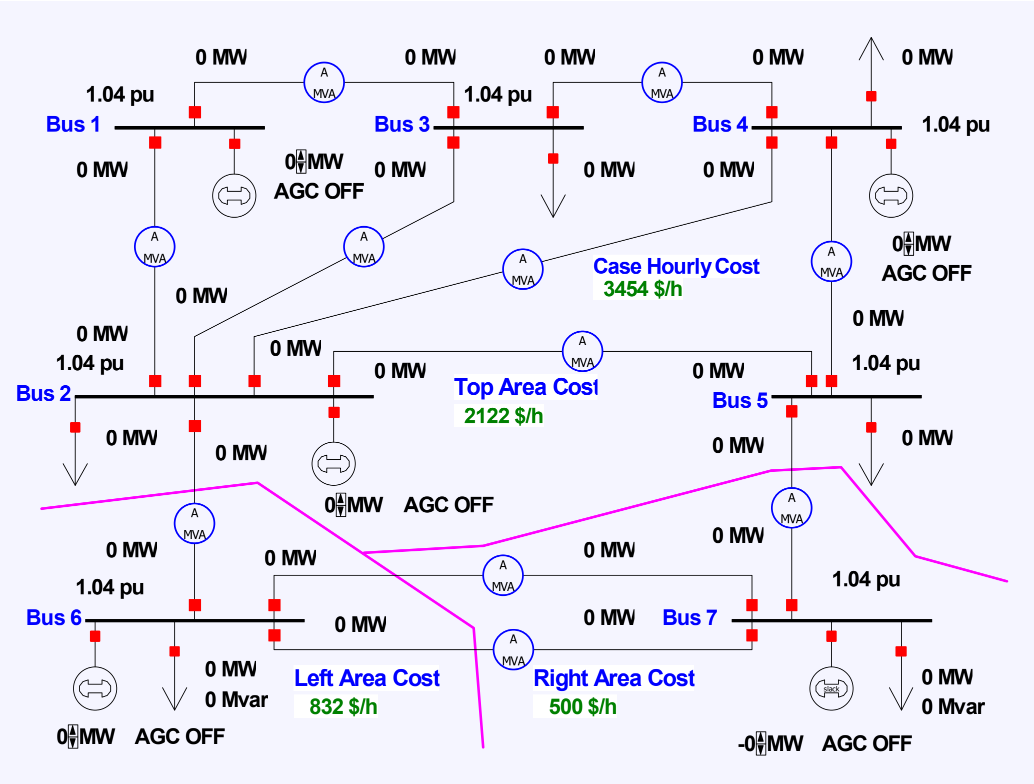
$$\left(I_s - Y_{se} Y_{ee}^{-1} I_e\right) = \left(Y_{ss} - Y_{se} Y_{ee}^{-1} Y_{es}\right) V_s$$

Ward Type Equivalencing Considerations

- At the end of the Ward process often many of the new equivalent lines have high impedances
  - Often there is an impedance threshold, and lines with impedances above this value are eliminated
- The equivalent lines may have unusual values, including negative resistances
- Load and generation is represented as equivalent current injections or shunts; sometimes these values are converted back to constant power
- Consideration needs to be given to loss of reactive support; the equivalent embeds the present load and generation values

B7Flat_Eqv Example

- In this example the B7Flat_Eqv case is reduced, eliminating buses 1, 3 and 4. The study system is then 2, 5, 6, 7, with buses 2 and 5 the boundary buses



For ease of comparison system is modeled unloaded

B7Flat_Eqv Example

- Original $\mathbf{Y}_{\text{bus}}$

$$\mathbf{Y}_{bus} = j\begin{bmatrix} -20.83 & 16.67 & 4.17 & 0 & 0 & 0 & 0 \\ 16.67 & -52.78 & 5.56 & 5.56 & 8.33 & 16.67 & 0 \\ 4.17 & 5.56 & -43.1 & 33.3 & 0 & 0 & 0 \\ 0 & 5.56 & 33.3 & -43.1 & 4.17 & 0 & 0 \\ 0 & 8.33 & 0 & 4.17 & -29.17 & 0 & 16.67 \\ 0 & 16.67 & 0 & 0 & 0 & -25 & 8.33 \\ 0 & 0 & 0 & 0 & 16.67 & 8.33 & -25 \end{bmatrix}$$

$$\mathbf{Y}_{ee} = j\begin{bmatrix} -20.833 & 4.167 & 0 \\ 4.167 & -43.056 & 33.333 \\ 0 & 33.333 & -43.056 \end{bmatrix}$$

B7Flat_Eqv Example

$$\mathbf{Y}_{es} = j\begin{bmatrix} 16.667 & 0 & 0 & 0 \\ 5.556 & 0 & 0 & 0 \\ 5.556 & 4.167 & 0 & 0 \end{bmatrix} \quad \mathbf{Y}_{se} = j\begin{bmatrix} 16.667 & 5.556 & 5.556 \\ 0 & 0 & 4.167 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

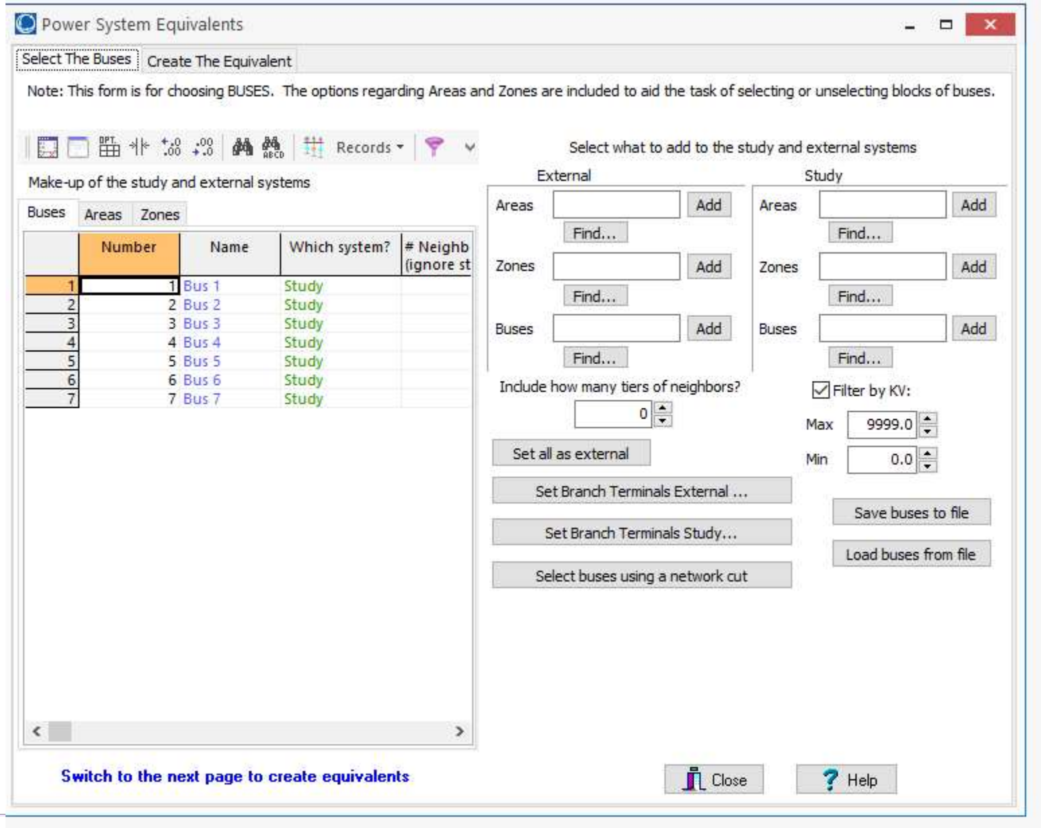
$$\mathbf{Y}_{ss} = j\begin{bmatrix} -52.778 & 8.333 & 16.667 & 0 \\ 8.333 & -29.167 & 0 & 16.667 \\ 16.667 & 0 & -25.0 & 8.333 \\ 0 & 16.667 & 8.333 & -25.0 \end{bmatrix}$$

Note $\mathbf{Y}_{es}$=$\mathbf{Y}_{se}$' if no phase shifters

$$\left(\mathbf{Y}_{ss} - \mathbf{Y}_{se}\mathbf{Y}_{ee}^{-1}\mathbf{Y}_{es}\right) = j\begin{bmatrix} -28.128 & 11.463 & 16.667 & 0 \\ 11.463 & -28.130 & 0 & 16.667 \\ 16.667 & 0 & -25.0 & 8.333 \\ 0 & 16.667 & 8.333 & -25.0 \end{bmatrix}$$

Equivalencing in PowerWorld

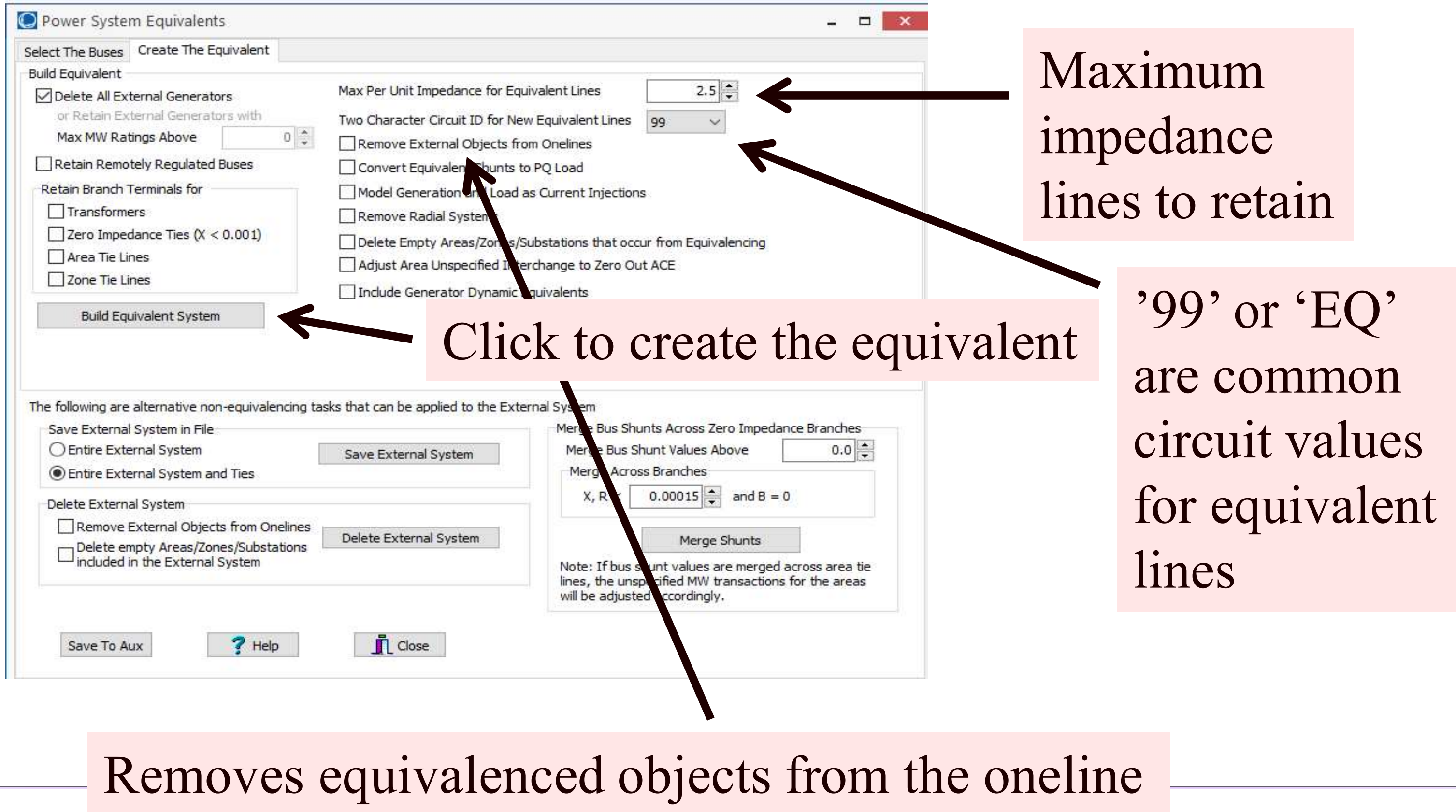
- Open a case and solve it; then select **Edit Mode**, **Tools**, **Equivalencing;** this displays the Power System Equivalents Form



Next step is then to divide the buses into the study system and the external system; buses can be loaded from a text file as well

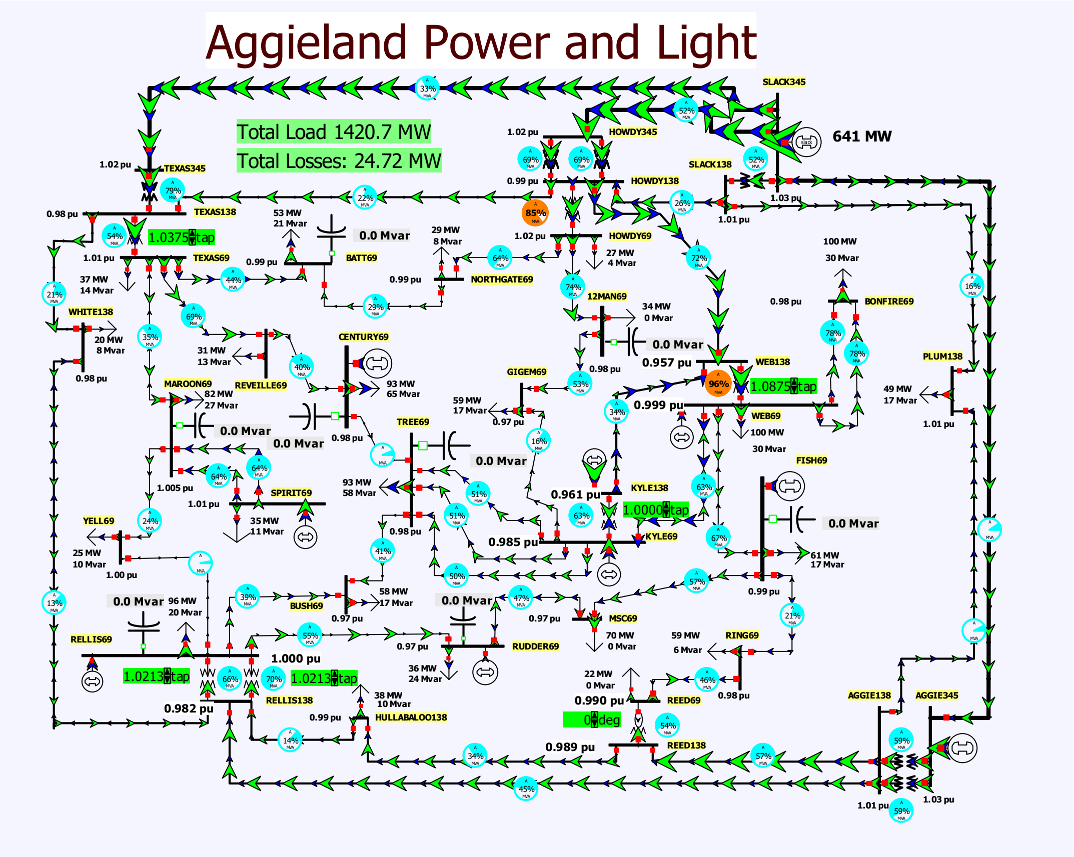
Equivalencing in PowerWorld

- Then go to the **Create The Equivalent** page, select the desired options and select **Build Equivalent System**



Small System Equivalent Example

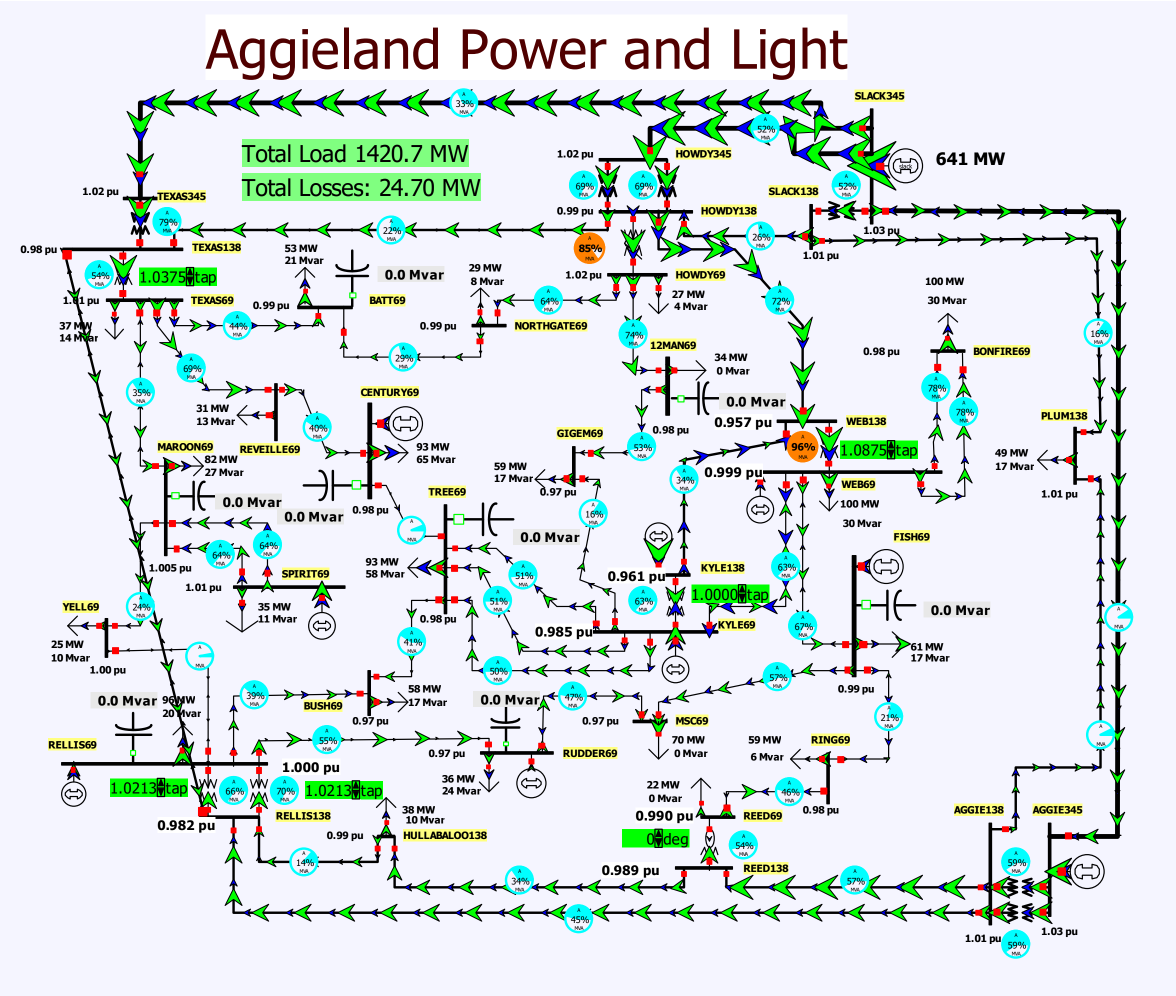
- Example shows the creation of an equivalent for Aggieland37 example



First example is simple, just removing WHITE138 (bus 3); note TEXAS138 is now directly joined to RELLIS138..

Case is Aggieland37

Small System Equivalent Example

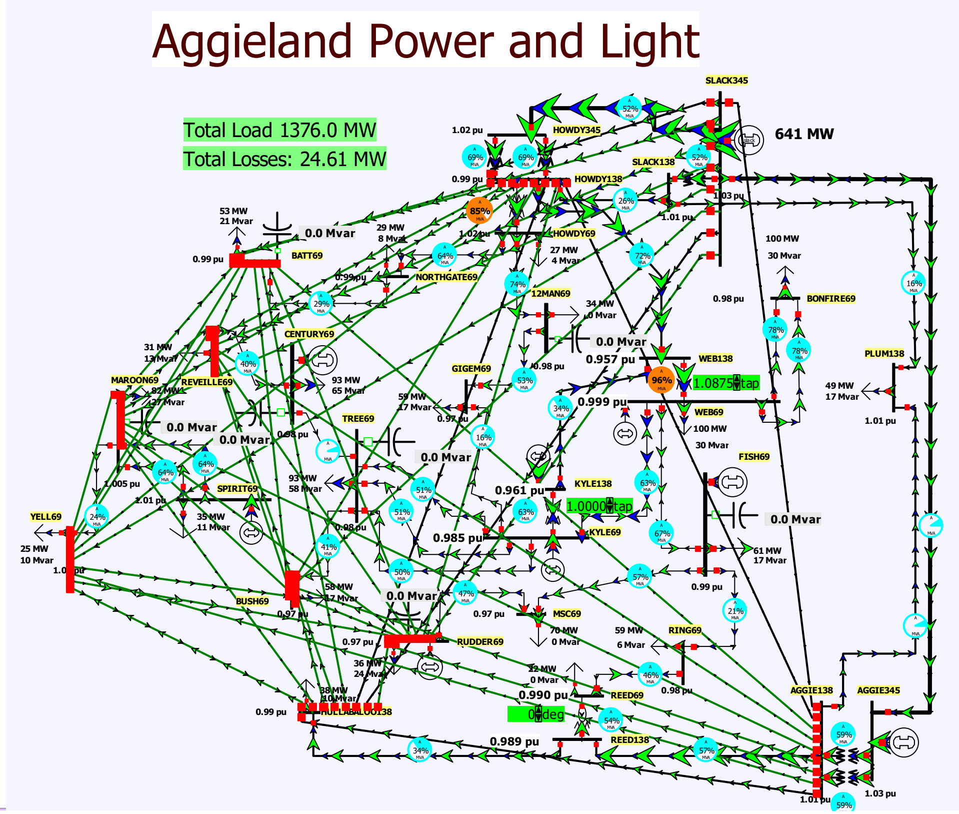


Only bus 3 was removed; the new equivalent line was auto-inserted.

Don’t save the equivalent with the same name as the original, unless you want to lose the original

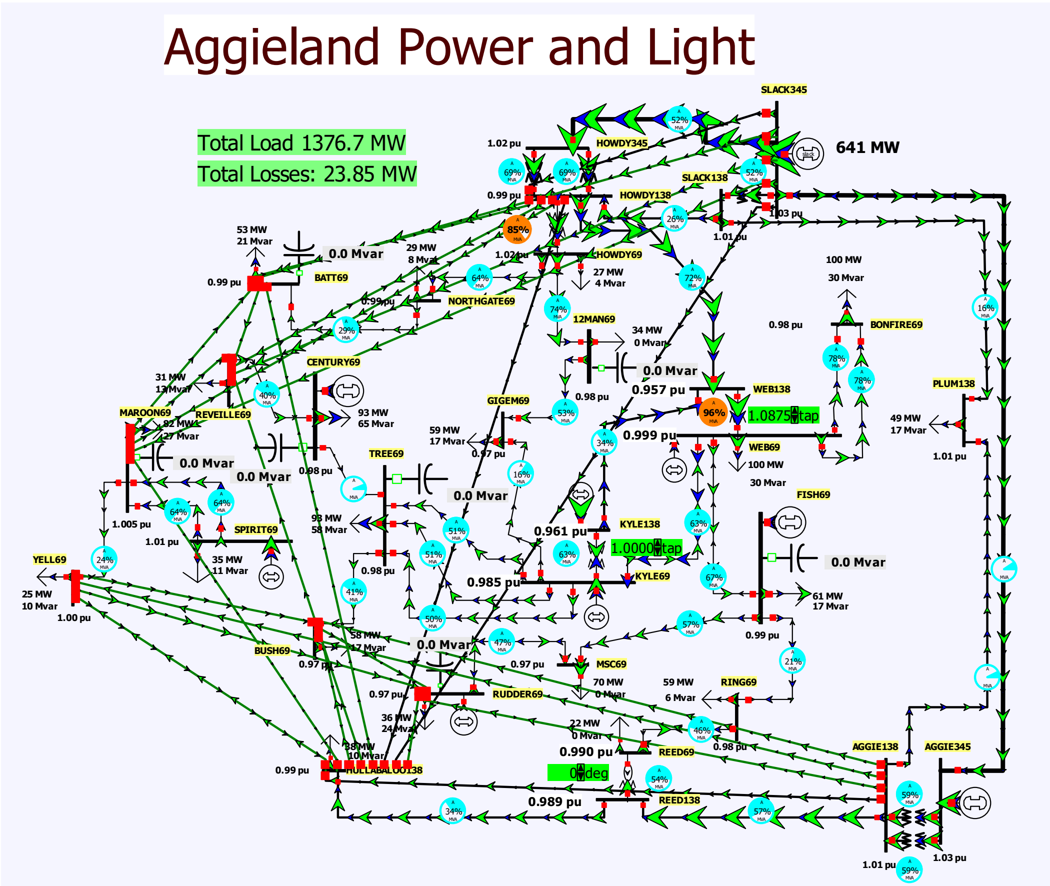
Small System Equivalent Example

- Now remove buses at WHITE138 and TEXAS and RELLIS (1, 3, 12, 40, 41, 44); set **Max Per Unit Impedance for Equivalent Lines** to 99 (per unit) to retain all lines. Again use the auto-insert to show the equivalent lines.



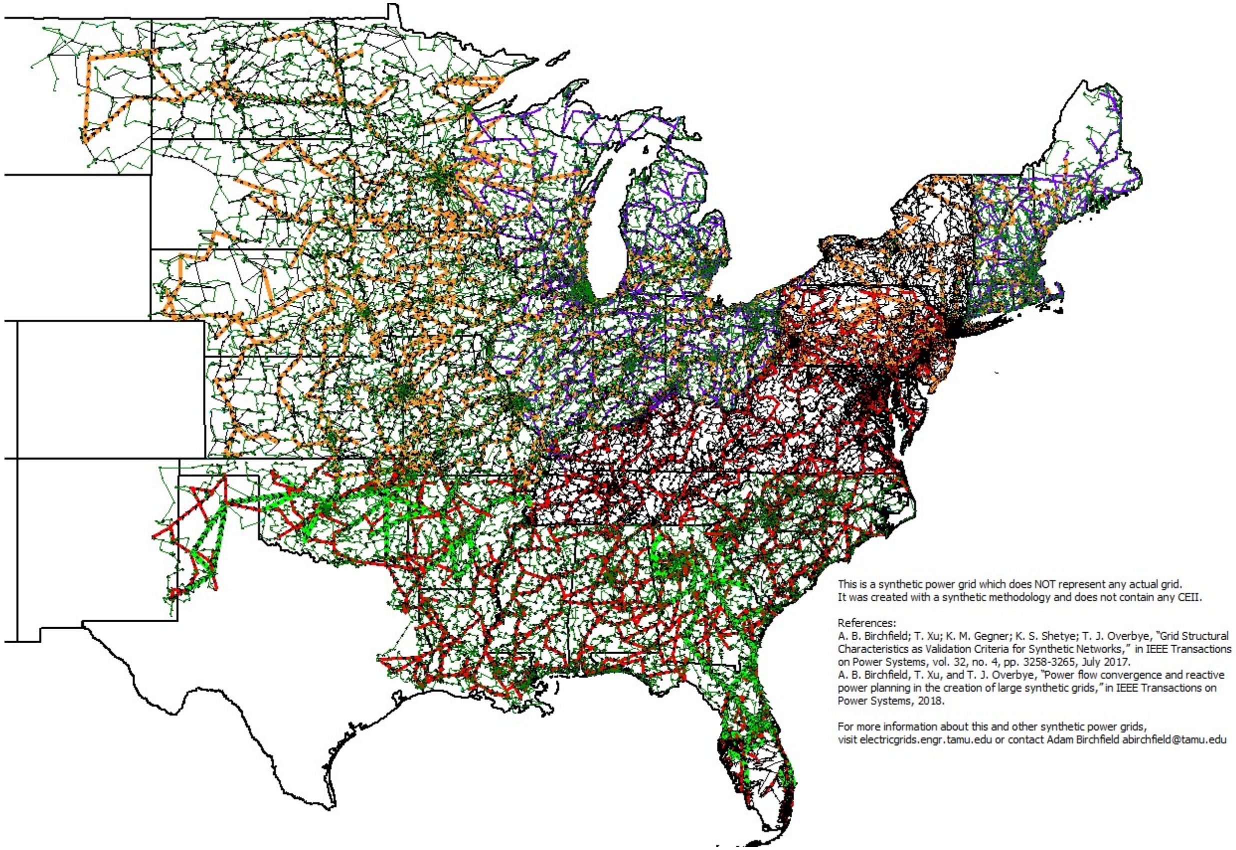
Small System Equivalent Example

- Now set the **Max Per Unit Impedance for Equivalent Lines** to 2.5.



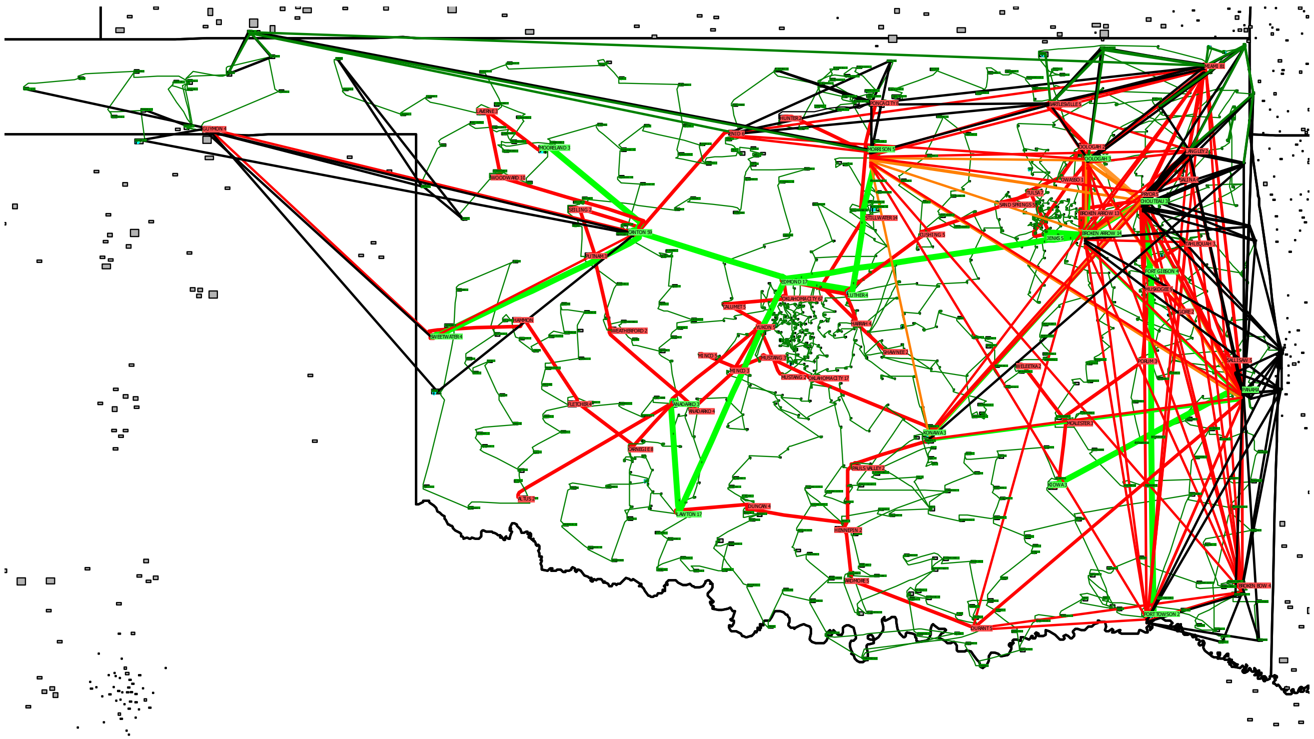
Large System Example: 70K Case

- Original System has 70,000 buses and 71,343 lines



Large System Example: 70K Case

- Just retain the Oklahoma Area; now 1591 buses and 1745 lines (deleting ones above 2.5 pu impedance)



When equivalencing so many buses it is best to first close the oneline. Then reopen the oneline and select **Onelines, List Display, Unlinked Display Objects** and use the button on this display to delete them.

Grid Equivalent Examples

- A 2016 EI case had about 350 lines with a circuit ID of '99' and about 60 with 'EQ' (out of a total of 102,000)
  - Both WECC and the EI use '99' or 'EQ' circuit IDs to indicate equivalent lines
  - One would expect few equivalent lines in interconnect wide models
- A 12 year old EI case had about 1633 lines with a circuit ID of '99' and 400 with 'EQ' (out of a total of 65673)
- A 12 year old case with about 5000 buses and 5000 lines had 600 equivalent lines