ECEN 615
Methods of Electric Power Systems Analysis

Lecture 9: Sparse System

Prof. Tom Overbye
Dept. of Electrical and Computer Engineering
Texas A&M University
overbye@tamu.edu

Announcements

- Start reading Chapter 7 (the term reliability is now often used instead of security)
- Homework 2 is due today
- Homework 3 is due on Thursday Sept 29

Sparse Factorization Example

- For a second example, again consider the same system, except with the nodes renumbered

$$\mathbf{B} = \begin{bmatrix} 10 & -4 & -3 & -2 \\ -4 & 5 & 0 & 0 \\ -3 & 0 & 4 & 0 \\ -2 & 0 & 0 & 3 \end{bmatrix}$$

Sparse Factorization Examples, Cont.

- With i=2, load SWR = [-4 5 0 0]
  - p2 = B[2,1]
  - p1 = B[1,1]
  - SWR[1]=-4/p1.value=-4/10 = -0.4
  - p1 = B[1,2]
  - SWR[2]=5 – (-0.4)*(-4) = 1.6
  - p1 = B[1,3]
  - SWR[3]= 0 – (-0.4)*(-3) = -1.2
  - p1 = B[1,4]
  - SWR[4]=0 – (-0.4)*(-2) = -0.8
  - p2=p2.next=diag so done
  - UnloadSWR and **we have a problem!**

$$\mathbf{B} = \begin{bmatrix} 10 & -4 & -3 & -2 \\ -4 & 5 & 0 & 0 \\ -3 & 0 & 4 & 0 \\ -2 & 0 & 0 & 3 \end{bmatrix}$$

There are no elements in row 2 for columns 3 and 4!

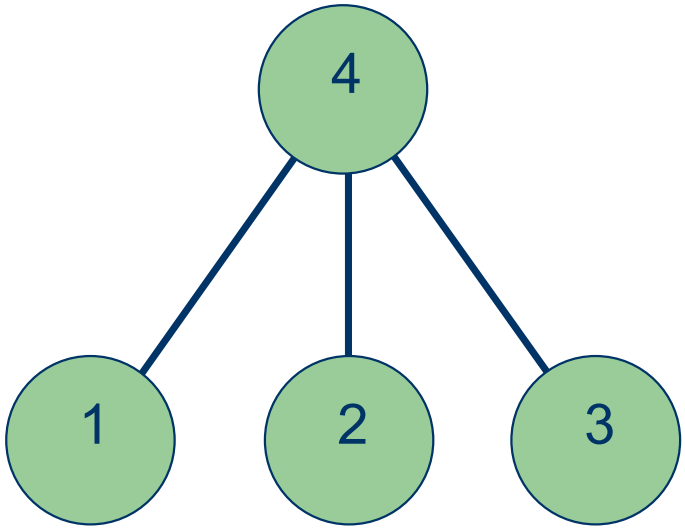
Fills

- When doing a factorization of a sparse matrix some values that were originally zero can become nonzero during the factorization process
- These new values are called “fills”
  (some call them fill-ins)
- For a structurally symmetric matrix the fill occurs for both the element and its transpose value (i.e., $\mathbf{A}_{ij}$ and $\mathbf{A}_{ji}$)
- How many fills are required depends on how the matrix is ordered
  - For a power system case this depends on the bus ordering

Fills

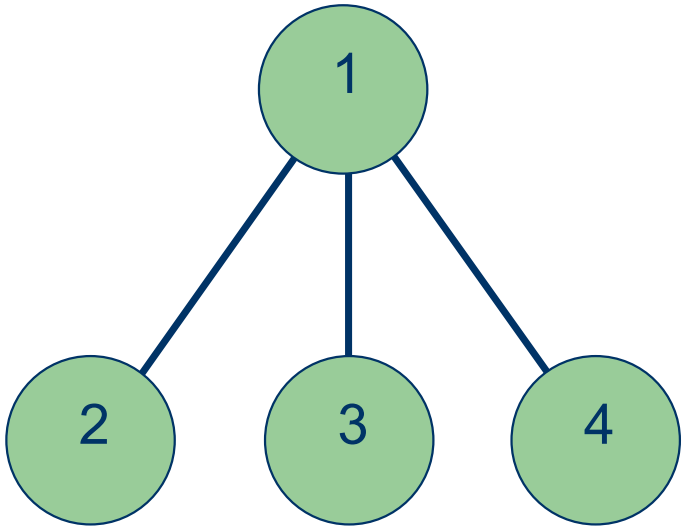
- There are two key issues associated with fills
  - Adding the fills
  - Ordering the matrix elements (buses in our case) to reduce the number of fills
- The amount of computation required to factor a sparse matrix depends upon the number of nonzeros in the original matrix, and the number of fills added
- How the matrix is ordered can have a dramatic impact on the number of fills, and hence the required computation
- Usually a matrix cannot be ordered to totally eliminate fills

Fill Examples



$$\mathbf{A} = \begin{bmatrix} 5 & 0 & 0 & -4 \\ 0 & 4 & 0 & -3 \\ 0 & 0 & 3 & -2 \\ -4 & -3 & -2 & 10 \end{bmatrix}$$

No Fills Required



$$\mathbf{B} = \begin{bmatrix} 10 & -4 & -3 & -2 \\ -4 & 5 & 0 & 0 \\ -3 & 0 & 4 & 0 \\ -2 & 0 & 0 & 3 \end{bmatrix}$$

Fills Required (matrix becomes full)

Example: 7 by 7 Matrix

- Consider the 7 x 7 matrix **A** with the zero-nonzero pattern shown on the left: of the 49 possible elements there are only 31 that are nonzero
- If elimination proceeds with the given ordering, all but two of the 18 originally zero entries, will fill in, as seen in the right image

The original zero-nonzero structure

r \ c	1	2	3	4	5	6	7
1	X	X	X	X	X	X	
2	X	X	X			X	X
3	X	X	X			X	X
4	X			X	X		
5	X			X	X	X	
6	X	X	X		X	X	
7		X	X				X

r \ c	1	2	3	4	5	6	7
1	X	X	X	X	X	X	
2	X	X	X	F	F	X	X
3	X	X	X	F	F	X	X
4	X	F	F	X	X	F	F
5	X	F	F	X	X	X	F
6	X	X	X	F	X	X	F
7		X	X	F	F	F	X

The post-elimination zero nonzero pattern

Example: 7 by 7 Matrix Reordering

- We next reorder the rows and the columns of **A** so as to result in the pattern shown in (c)
- For this reordering, we obtain no fills, as shown in the table of factors given in (d)
- In this way, we preserve the original sparsity of **A**

r \ c	4	5	1	6	7	3	2
4	X	X	X				
5	X	X	X	X			
1	X	X	X	X		X	X
6		X	X	X		X	X
7					X	X	X
3			X	X	X	X	X
2			X	X	X	X	X

The reordered system

r \ c	4	5	1	6	7	3	2
4	X	X	X				
5	X	X	X	X			
1	X	X	X	X		X	X
6		X	X	X		X	X
7					X	X	X
3			X	X	X	X	X
2			X	X	X	X	X

The post- elimination reordered system

Graph (Grid) Insights

- For electric grids it can be helpful to relate the sparse matrix back to an associated electric grid
- Assume an n by n matrix A
- All the diagonals are non-zeros
- If there is a connection between two buses (nodes), say at position j and k, then the associated matrix entries, $\mathbf{A}_{jk}$ and $\mathbf{A}_{kj}$ are nonzero

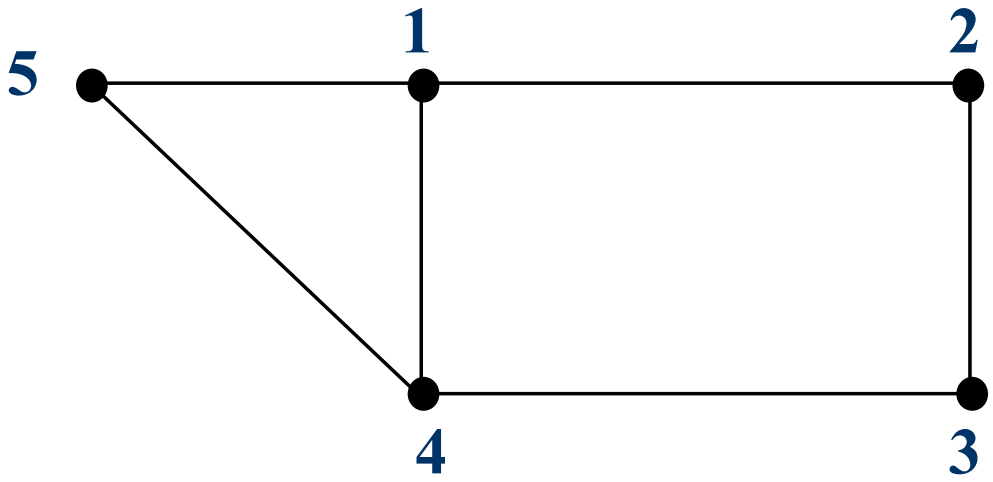
Example: 5 by 5 System

- Suppose that **A** has the zero-nonzero pattern

r \ *c*	1	2	3	4	5
1	*X*	*X*		*X*	*X*
2	*X*	*X*	*X*		
3		*X*	*X*	*X*	
4	*X*		*X*	*X*	*X*
5	*X*			*X*	*X*

Example: 5 by 5 System

- Then, the associated graph G is



We could also go from the graph to the matrix

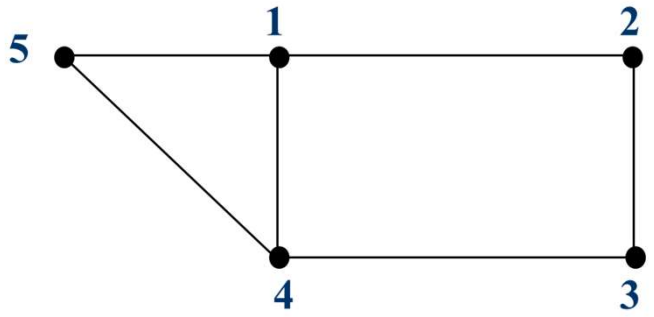
Graph-Theoretic Interpretation

- The graph-theoretic interpretation for the ordering and fill addition of the node (bus) j is as follows
  - As bus j is ordered it is deleted from the graph.
  - The deletion of the bus j involves all its incident branches
  - In the pre-elimination graph of the eliminated bus j, the elimination of the branches (j, k) and (l, j) results in the addition of the new branch (k, l), if one does not already exist

Example: 5 by 5 System

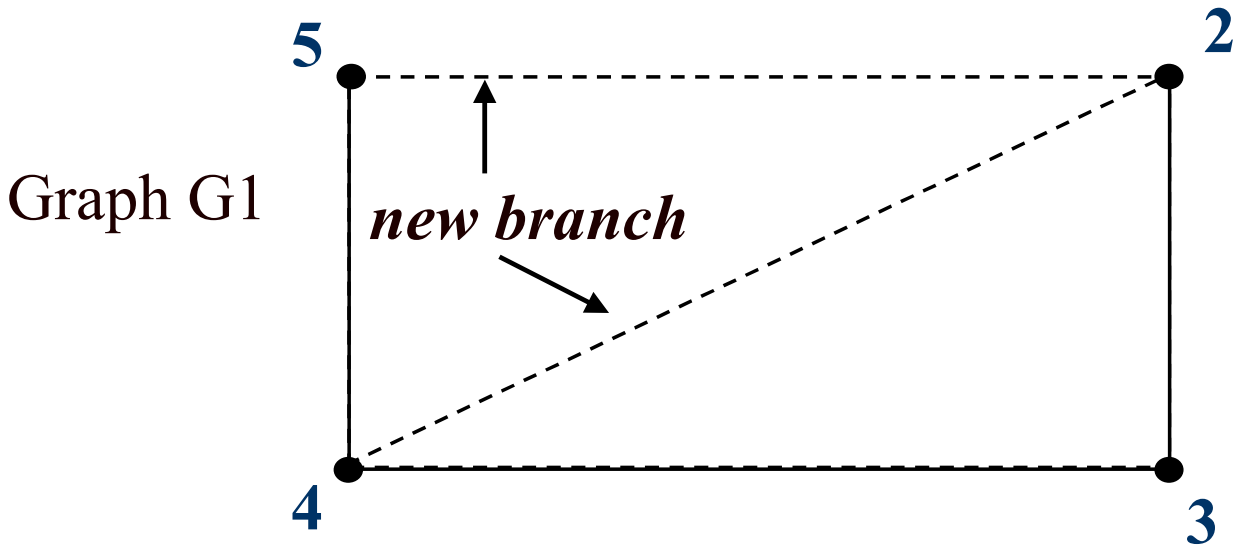
- If we decide to order bus 1, then it is deleted from the graph, with conditions added between its incident buses (2, 4, 5)

r \ c	1	2	3	4	5
1	X	X		X	X
2	X	X	X	F	F
3		X	X	X	
4	X	F	X	X	X
5	X	F		X	X



Here new lines are added between 2 and 4, and between 2 and 5

Example: 5 by 5 System



- We obtain the graph G1 from G by removing Bus 1 with the new added branches (2, 4) and (2, 5) corresponding to the fills

Example: 5 by 5 System

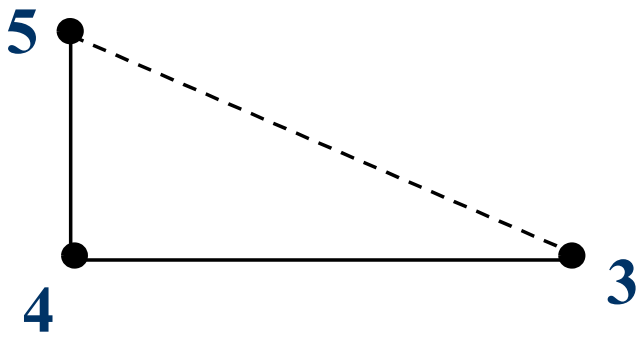
- The elimination of Bus 2 results in the submatrix shown below

r \ c	3	4	5
3	*X*	*X*	*F*
4	*X*	*X*	*X*
5	*F*	*X*	*X*

Now a new branch is added between buses 3 and 5

Example: 5 by 5 System

with the corresponding graph G2



- The elimination of Bus 3 yields

r \ c	4	5
4	X	X
5	X	X

Example: 5 by 5 System

with the corresponding graph G3

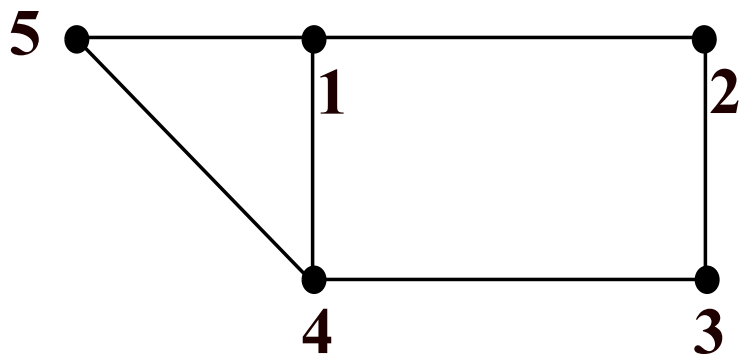
- Finally, upon Bus 4 we have

r \ c	5
5	X

- and the corresponding G4 is simply the point

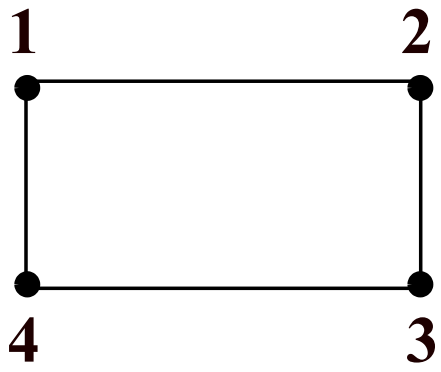
Reording the Rows/Columns

- We next examine how we may reorder the rows and columns of **A** to preserve its sparsity, i.e., to minimize the number of fills
- Eventually we'll introduce an algorithm to try to minimize the fills
- This is motivated by revisiting the graph G



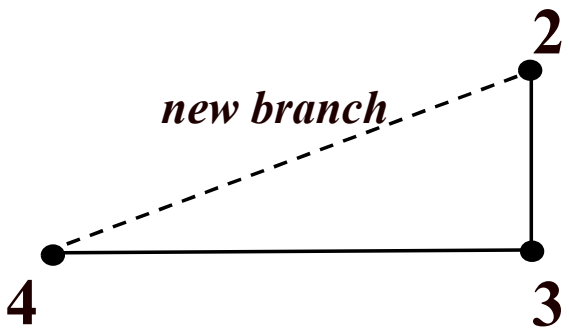
Reording Motivating Example

- To minimize the number of fills, i.e., the number of new branches in G, we eliminate first the node which upon deletion introduces the least number of new branches
- This is node 5 and upon deletion no new branches are added and the resulting graph G1 is



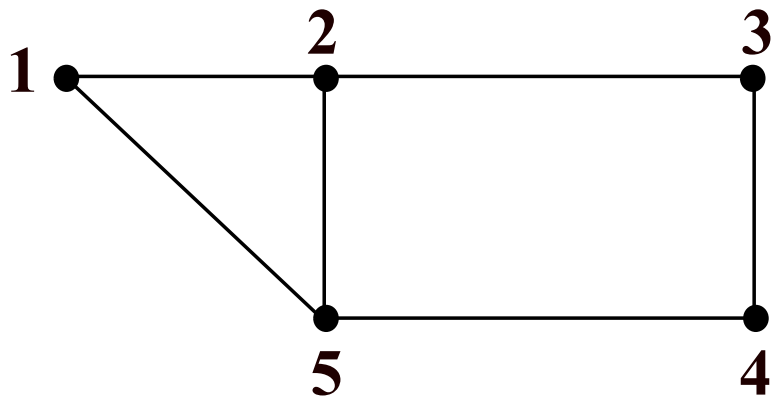
Reording Motivating Example

- The structure of G1 is such that any one of the remaining nodes may be chosen as the next node to be eliminated since each of the 4 remaining nodes introduces a new branch after its elimination
- We arbitrarily pick node 1 and we obtain the graph G2
- We continue with the next three choices arbitrary, resulting in no new fills



Reording Motivating Example

- We may relabel the original graph in such a way that the label of the node refers to the order in which it is eliminated
- Thus we renumber the nodes as shown below



Reording Motivating Example

- Clearly, relabeling the nodes corresponds to reordering the rows and columns of **A**
- For the reordered system, the zero-nonzero pattern of **A** is

r \ c	1	2	3	4	5
1	X	X			X
2	X	X	X		X
3		X	X	X	
4			X	X	X
5	X	X		X	X

Reording Motivating Example

and of its table of factors has the zero-nonzero structure

r \ c	1	2	3	4	5
1	X	X			X
2	X	X	X		X
3		X	X	X	F
4			X	X	X
5	X	X	F	X	X

Compared to the original ordering scheme, the new ordering scheme has saved us 4 fill-ins

General Findings

- The associated graph of the structurally symmetric matrix **A** is useful in gaining insights into the factorization process
- We make the following observations
  - If **A** is originally structurally symmetric, then it remains so in all the steps of the factorization;
  - A good ordering scheme is independent of the values of the elements of **A** and depends only on its the zero-nonzero pattern

Permutation Vectors

- Often the matrix itself is not physically reorded when it is renumbered. Rather we can make use of what is known as a permutation vector, and (if needed) an inverse permutation vector
- These vectors implement the following functions
  - $i_{new} = New(i_{old})$
  - $i_{old} = Old(i_{new})$
- For an n by n matrix the permutation vector is an n-sized integer vector
- If ordered lists are needed, then the linked lists do need to be reordered, but this can be done quickly

Permutation Vectors, cont.

- For the previous five bus example, in which the buses are to be reordered to (5,1,2,3,4), the permutation vector would be **rowPerm**=[5,1,2,3,4]
  - That is, the first row to consider is row 5, then row 1, …
- If needed, the inverse permutation vector is **invRowPerm** = [2,3,4,5,1]
  - That is, with the reordering the first element is in position 2, the second element in position 2, ….
- Hence i = invRowPerm[rowPerm[i]]

Sparse Factorization using a Permutation Vector

```
For i := 1 to n Do Begin
  k = rowPerm[i];  // this is the only change, except using k
  LoadSWRbyCol(k,SWR);   // Load Sparse Working Row }
  p2 := rowHead[k];  // the row needs to be ordered correctly!
  While p2 <> rowDiag[k] Do Begin
    p1 := rowDiag[p2.col];
    SWR[p2.col] := SWR[p2.col] / p1.value;
    p1 := p1.next;
    While p1 <> nil Do Begin   // Go to the end of the row
      SWR[p1.col] := SWR[p1.col] - SWR[p2.col] *p1.value;
      p1 := p1.next;
    End;
    p2 := p2.next;
  End;
  UnloadSWRByCol(k,SWR);
End;
```

Sparse Matrix Reordering

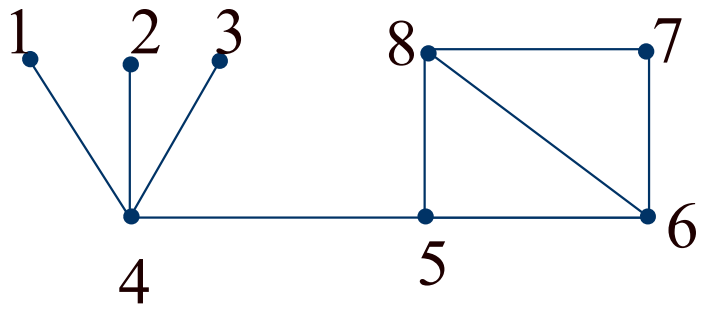
- There is no computationally efficient way to optimally reorder a sparse matrix; however there are very efficient algorithms to greatly reduce the fills
- Two steps here: 1) order the matrix, 2) add fills
- A quite common algorithm combines ordering the matrix with adding the fills
- The two methods discussed here were presented in the 1963 paper by Sato and Tinney from BPA; known as Tinney Scheme 1 and Tinney Scheme 2 since they are more explicitly described in Tinney's 1967 paper
  - 1967 paper also has Tinney Scheme 3 (briefly covered)

Tinney Scheme 1

- Easy to describe, but not really used since the number of fills, while reduced, is still quite high
- In graph theory the degree (or valence or valency) of a vertex is the number of edges incident to the vertex
- Order the nodes (buses) by the number of incident branches (i.e., its valence) those with the lowest valence are ordered first
  - Nodes with just one incident line result in no new fills
  - Obviously in a large system many nodes will have the same number of incident branches; ties can be handled arbitrarily

Tinney Scheme 1, Cont.

- Once the nodes are reordered, the fills are added
  - Common approach to ties is to take the lower numbered node first
- A shortcoming of this method is as the fills are added the valence of the adjacent nodes changes



Tinney 1 order is 1,2,3,7,5,6,8,4

Node	Valence
1	1
2	1
3	1
4	4
5	3
6	3
7	2
8	3

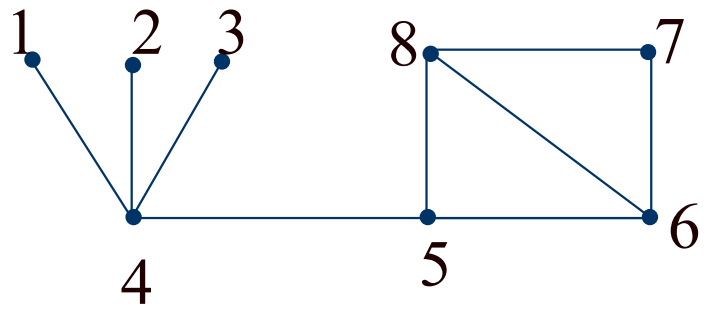
Number of new branches is 2 (4-8, 4-6)

Tinney Scheme 2

- The Tinney Scheme 2 usually combines adding the fills with the ordering in order to update the valence on-the-fly as the fills are added
- As before the nodes are chosen based on their valence, but now the valence is the actual valence they have with the added lines (fills)
  - This is also known as the Minimum Degree Algorithm (MDA)
  - Ties are again broken using the lowest node number
- This method is quite effective for power systems, and is highly recommended; however it is certainly not guaranteed to result in the fewest fills (i.e. not optimal)

Tinney Scheme 2 Example

- Consider the previous network:



- Nodes 1,2,3 are chosen as before. But once these nodes are eliminated the valence of 4 is 1, so it is chosen next. Then 5 (with a new valence of 2 tied with 7), followed by 6 (new valence of 2), 7 then 8.

Coding Tinney 2

- The following slides show how to code Tinney 2 for an n by n sparse matrix **A**
- First we setup linked lists grouping all the nodes by their original valence
- vcHead is a pointer vector [0..mvValence]
  - If a node has no connections its incidence is 0
  - Theoretically mvValence should be n-1, but in practice a much smaller number can be used, putting nodes with valence values above this into the vcHead[mvValence] is

Coding Tinney 2, cont.

- Setup a boolean vectors chosenNode[1..n] to indicate which nodes are chosen and BSWR[1..n] as a sparse working row; initialize both to all false
- Setup an integer vector rowPerm[1..n] to hold the permuted rows; initialize to all zeros
- For i := 1 to n Do Begin
  - Choose node from valence data structure with the lowest current valence; let this be node k
    - Go through vcHead from lastchosen level (last chosen level may need to be reduced by one during the following elimination process;
  - Set rowPerm[i] = k; set chosenNode[k] = true

Coding Tinney 2, cont.

- Modify sparse matrix **A** to add fills between all of k's adjacent nodes provided
  1. a branch doesn't already exist
  2. both nodes have not already been chosen (their chosenNode entries are false)
  - These fills are added by going through each element in row k; for each element set the BSWR elements to true for the incident nodes; add fills if a connection does not already exist (this requires adding two new elements to **A**)
- Again go through row k updating the valence data structure for those nodes that have not yet been chosen
  - These values can either increase or go down by one (because of the elimination of node k)

- This continues through all the nodes; free all vectors except for rowPerm
- At this point in the algorithm the rowPerm vector contains the new ordering and matrix **A** has been modified so that all the fills have been added
  - The order of the rows in **A** has not been changed, and its columns are no longer sorted

Coding Tinney 2, cont.

- Sort the rows of **A** to match the order in rowPerm
  - Surprising sorting **A** is of computational order equal to the number of elements in **A**
    - Go through **A** putting its elements into column linked lists; these columns will be ordered by row
    - Then through the columns linked lists in reverse order given by rowPerm
      - That is For i := n downto 1 Do Begin
        p1 := TSparmatLL(colHead[rowPerm[i]).Head;
        ….
- That's it – the matrix **A** is now readying for factoring
- Pivoting may be required, but usually isn't needed in the power flow

Some Example Values for Tinney 2

Number of buses	Nonzeros before fills	Fills	Total nonzeros	Percent nonzeros
37	63	72	135	9.86%
118	478	168	646	4.64%
18,190	64,948	31,478	96,426	0.029%
62,605	228,513	201,546	430,059	0.011%

Tinney Scheme 3

- “Number the rows so that at each step of the process the next row to be operated upon is the one that will introduce the fewest new nonzero terms.”
- “If more than one row meets this criterion, select any one. This involves a trial simulation of every feasible alternative of the elimination process at each step. Input information is the same as for scheme 2).”
- Tinney 3 takes more computation and in general does not give fewer fills than the quicker Tinney 2
- Tinney got into the NAE in 1998

These are direct quotes from the Tinney-Walker 1967 IEEE Proceedings Paper

Sparse Forward Substitution with a Permutation Vector

Pass in **b** in bvector

```
For i := 1 to n Do Begin
  k = rowPerm[i];  // this is the only change, except using k
  p1 := rowHead[k];  // the row needs to be ordered correctly!
  While p1 <> rowDiag[k] Do Begin
    bvector[k] = bvector[k] – p1.value*bvector[p1.col];
    p1 := p1.next;
   End;
 End;
```

Sparse Backward Substitution with Permutation Vector

Pass in **b** in bvector

```
For i := n downto 1 Do Begin
  k = rowPerm[i];
  p1 := rowDiag[k].next;
  While p1 <> nil Do Begin
    bvector[k] = bvector[k] – p1.value*bvector[p1.col];
    p1 := p1.next;
   End;
   bvector[k] := bvector[k]/rowDiag[k].value;
End;
```

- Note, numeric problems such as matrix singularity are indicated with rowDiag[k].value being zero!

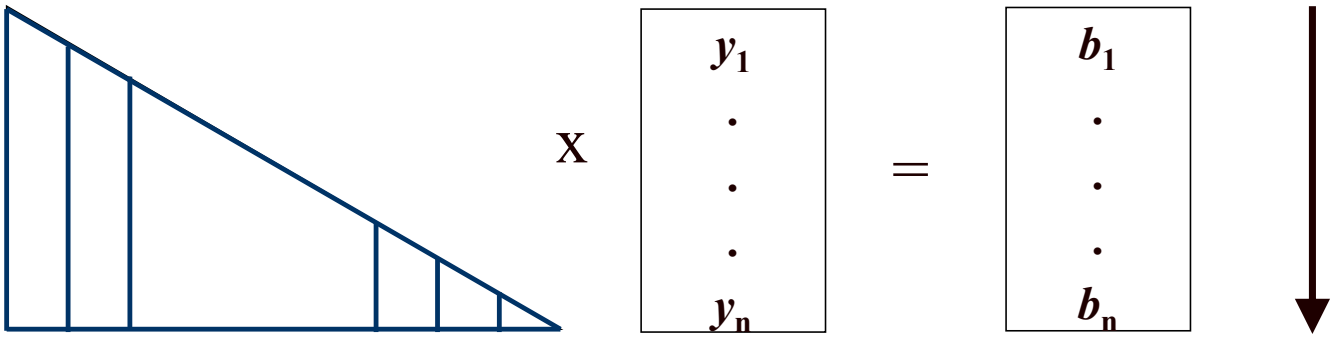
Sparse Vector Methods

- Sparse vector methods are useful for cases in solving **Ax**=**b** in which
  - **A** is sparse, **b** is sparse, only certain elements of **x** are needed
- In these right circumstances sparse vector methods can result in extremely fast solutions!
- A common example is to find selected elements of the inverse of **A**, such as diagonal elements.
- Often times multiple solutions with varying **b** values are required
  - **A** only needs to be factored once, with its factored form used many times
- Key reference is

W.F. Tinney, V. Brandwajn, and S.M. Chan, "Sparse Vector Methods", *IEEE Transactions on Power Apparatus and Systems*, vol. PAS-104, no. 2, February 1985, pp. 295-300

Sparse Vector Methods Introduced

- Assume we are solving **Ax = b** with **A** factored so we solve **LUx = b** by first doing the forward substitution to solve **Ly = b** and then the backward substitution to solve **Ux = y**
- A key insight: In the solution of **Ly = b** if **b** is sparse then only certain columns of **L** are required, and **y** is often sparse

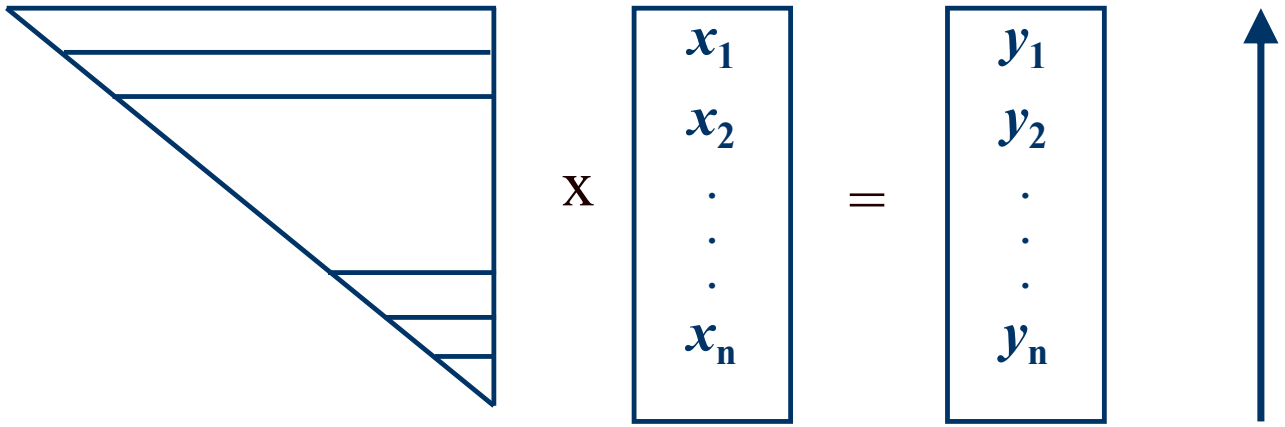


Fast Forward Substitution

- If **b** is sparse, then the fast forward (FF) substitution takes advantage of the fact that we only need certain columns of **L**
- We define {FF} as the set of columns of L needed for the solution of **Ly** = **b**; this is equal to the nonzero elements of **y**
- In general the solution of **Ux** = **y** will NOT result in **x** being a sparse vector
- However, oftentimes only certain elements of **x** are desired
  - E.g., the sensitivity of the flows on certain lines to a change in generation at a single bus; or a diagonal of $\mathbf{A}^{-1}$

Fast Backward Substitution

- In the case in which only certain elements of **x** are desired, then we only need to use certain rows in **U** below the desired elements of **x**; define these columns as {FB}
- This is known as a fast backward substitution (FB), which is used to replace the standard backward substitution



Factorization Paths

- We observe that
  - {FF} depends on the sparsity structures of **L** and **b**
  - {FB} depends on the sparsity structures of **U** and **x**
- The idea of the factorization path provides a systematic way to construct these sets
- A factorization path is an ordered set of nodes associated with the structure of the matrix
- For FF the factorization path provides an ordered list of the columns of **L**
- For FB the factorization path provides an ordered list of the rows of **U**

Factorization Paths

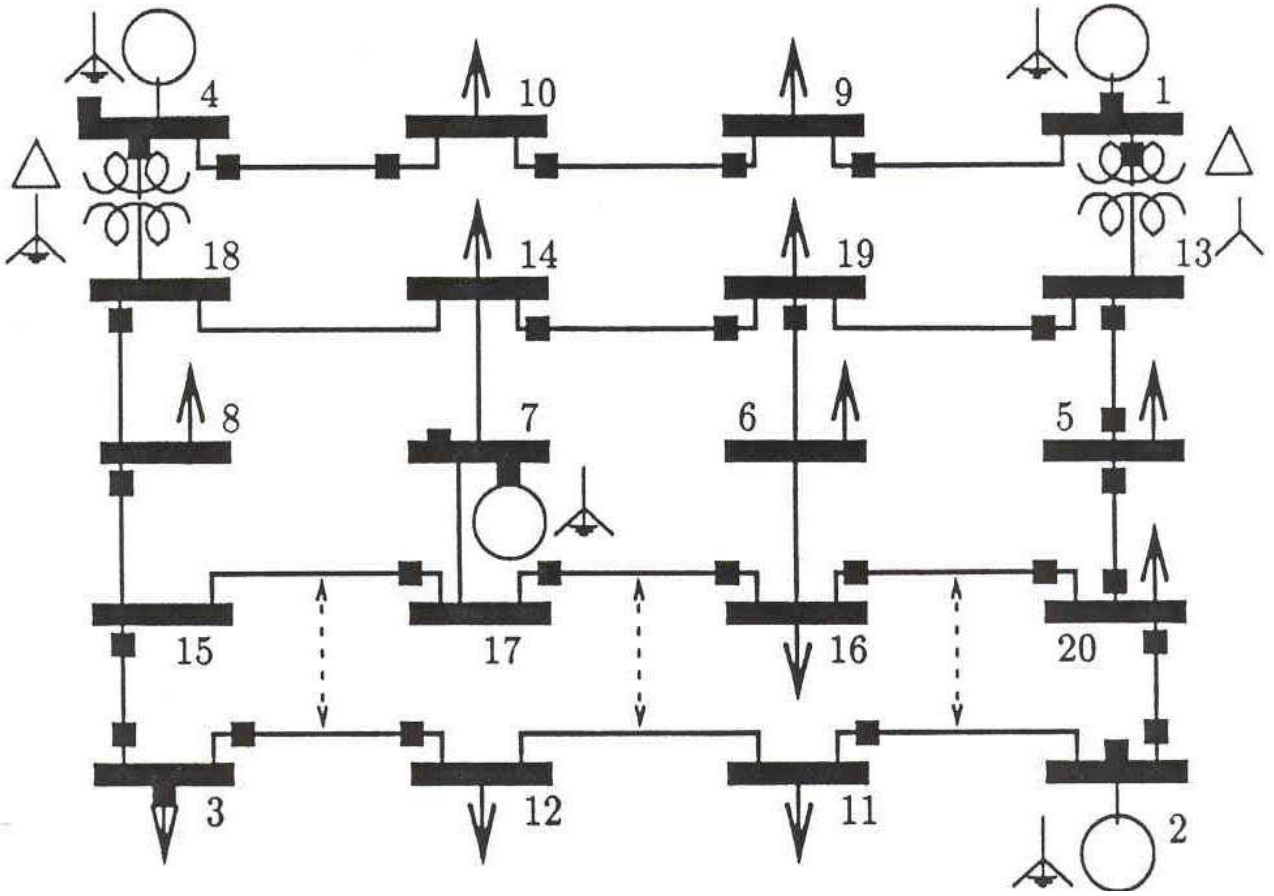
- The factorization path is traversed in the forward direction for FF and in the reverse direction for FB
  - Factorization paths should be built using doubly linked lists
- A singleton vector is a vector with just one nonzero element. If this value is equal to one then it is a unit vector as well.
- With a sparse matrix structure ordered based upon the permutation vector order the path for a singleton with a now zero at position arow is build using the following code:

```
p1:= rowDiag[arow];
 While p1 <> nil Do Begin
  AddToPath(p1.col);   // Setup a doubly linked list!
  p1 := rowDiag[p1.col].next;
 End;
```

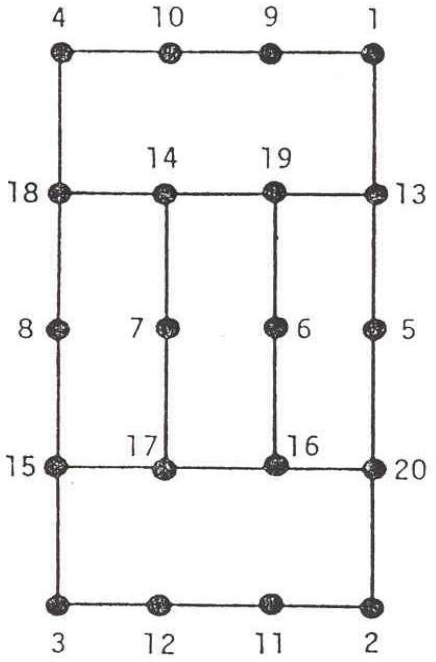
Path Table and Path Graph

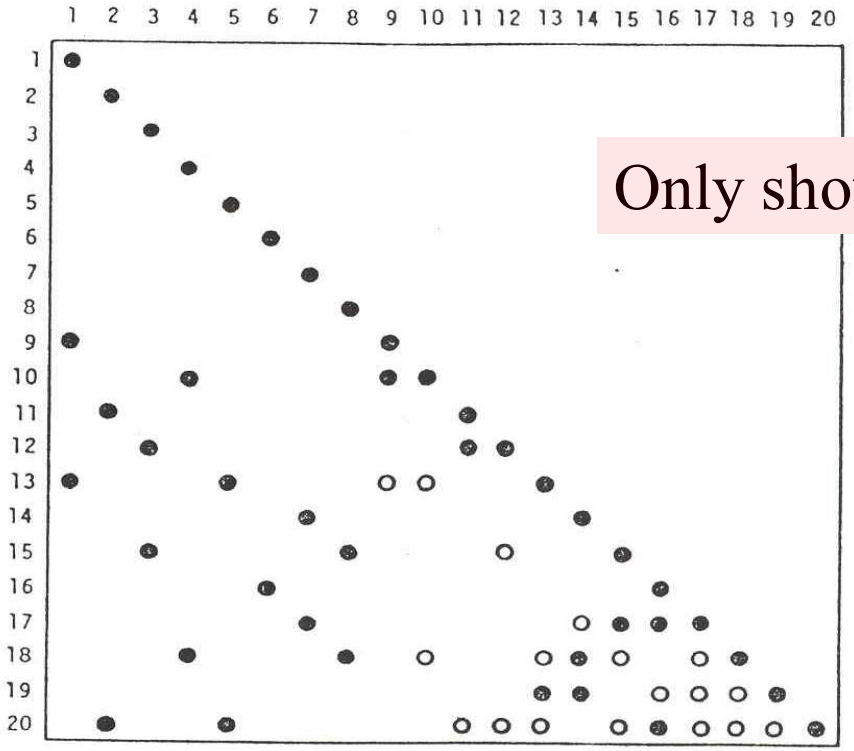
- The factorization path table is a vector that tells the next element in the factorization path for each row in the matrix
- The factorization path graph shows a pictorial view of the path table

20 Bus Example



20 Bus Example

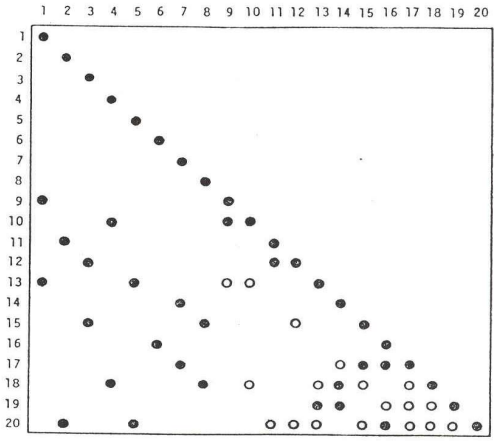




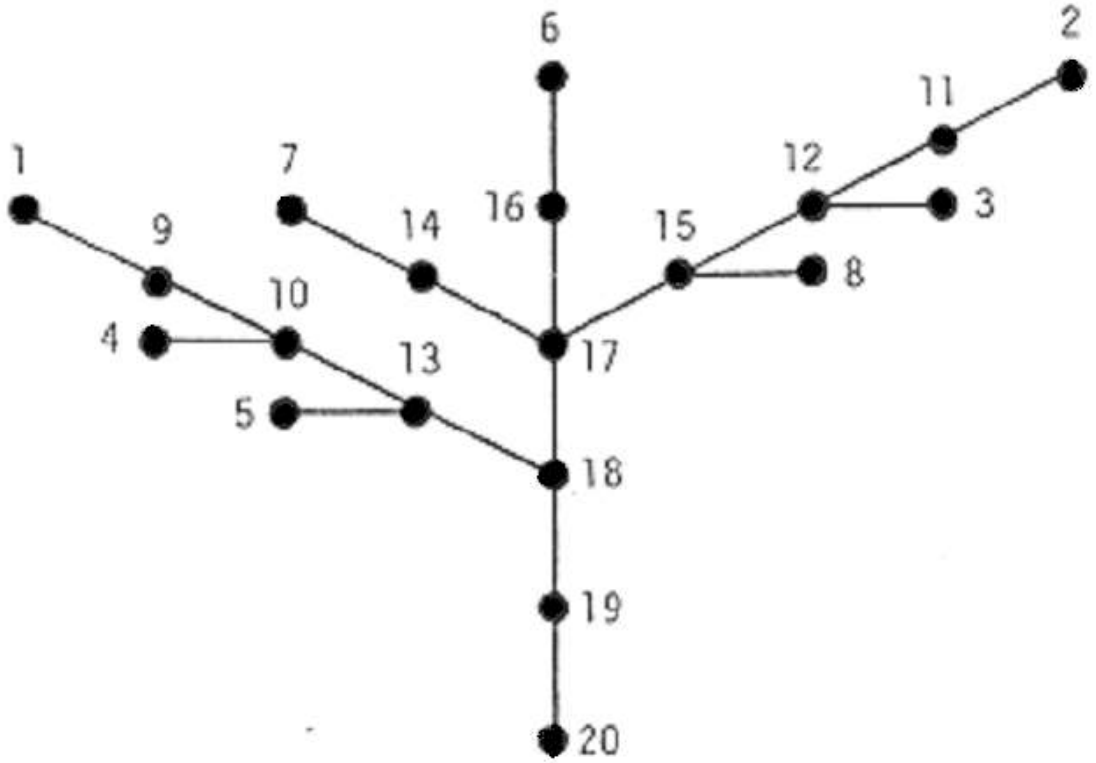
Only showing **L**

● Original Element (A and L)
○ Fill-In Element (L only)

20 Bus Example



NODE	NEXT	NODE	NEXT
1	9	11	12
2	11	12	15
3	12	13	18
4	10	14	17
5	13	15	17
6	16	16	17
7	14	17	18
8	15	18	19
9	10	19	20
10	13	20	0



20 Bus Example

- Suppose we wish to evaluate a sparse vector with the nonzero elements for components 2, 6, 7, and 12
- From the path table or path graph, we obtain the following factorization paths (f.p.)

$$\mathbf{2} \rightarrow \textit{f.p.}\ \{2, 11, 12, 15, 17, 18, 19, 20\}$$

$$\mathbf{6} \rightarrow \textit{f.p.}\ \{6, 16, 17, 18, 19, 20\}$$

$$\mathbf{7} \rightarrow \textit{f.p.}\ \{7, 14, 17, 18, 19, 20\}$$

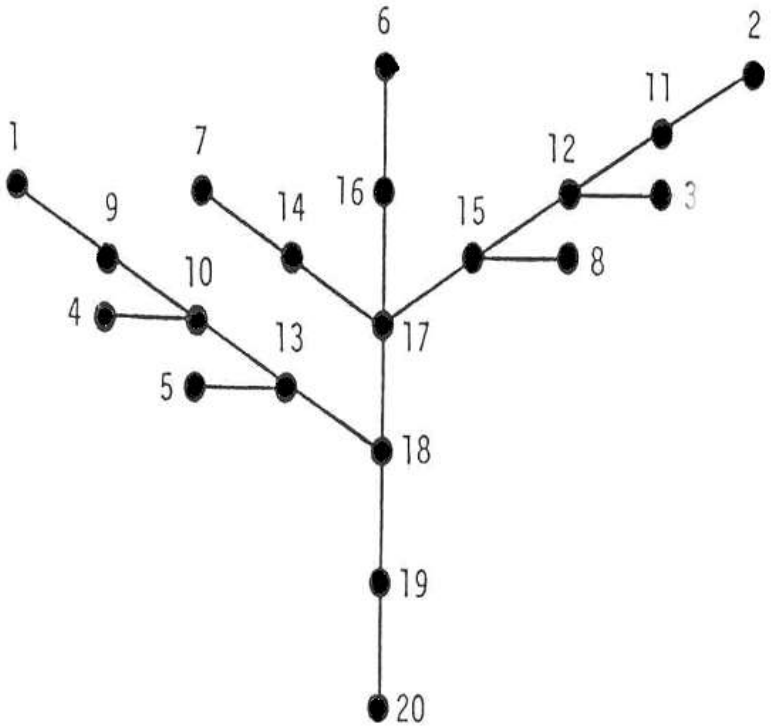
$$\mathbf{12} \rightarrow \textit{f.p. already contained in that for node } \mathbf{2}$$

- This gives the following path elements

$$\{7, 14, 6, 16, 2, 11, 12, 15, 17, 18, 19, 20\}$$

20 Bus Example

Full path



Desired subset