A Comparison of the Optimal Multiplier in Polar and Rectangular Coordinates

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Abstract—**Studies of the optimal multiplier (or optimal step size) modification to the standard Newton-Raphson load flow have mainly focused on highly stressed and unsolvable systems. This paper extends these previous studies by comparing performance of the Newton-Raphson load flow with and without optimal multipliers for a variety of unstressed, stressed, and unsolvable systems. Also, the impact of coordinate system choice in representing the voltage phasor at each bus is considered. In total, four solution methods are compared: the Newton-Raphson algorithm with and without optimal multipliers using polar and rectangular coordinates. This comparison is carried out by combining analysis of the optimal multiplier technique with empirical results for 2 bus, 118 bus, and 10,274 bus test cases. These results indicate that the polar Newton-Raphson load flow with optimal multipliers is the best method of solution for both solvable and unsolvable cases.**

Index Terms—**load flow analysis, load flow convergence, optimal multipliers, step size optimization, polar and rectangular coordinates.**

I. Introduction

THE increased utilization of existing transmission resources in the North American power grid without any significant upgrades has led to a system operated closer to the edge of reliability [1]. When performing contingency analysis on systems that are already operating very close to reliability limits, it is not uncommon to find system configurations where the system is either highly stressed or in an unsolvable situation [2], [3]. Attempting to solve such systems can lead to divergence in the standard Newton-Raphson (NR) load flow. Divergence can also occur when poor initial conditions are used to begin the solution process [4], or, in continuation load flows, when too large of a predictor step is taken [5].

Because divergence of the load flow can lead to wasteful computations and unpredictable behavior, significant research has been done to construct non-divergent algorithms for solving the load flow equations of highly stressed and unsolvable systems [6]-[10]. Reference [10] provides an excellent table summarizing several methods for stopping divergence of the load flow. The various methods used to prevent divergence of the NR load flow [11] have been shown to prevent divergence in many cases.

While non-divergence is an excellent characteristic to have in a load flow solution algorithm, any method used for load flow solutions must be both fast and robust for any type of system, whether the system is unstressed, stressed, or unsolvable. Unfortunately, little attention has been paid to the behavior of these non-divergent methods for normally convergent, unstressed systems in addition to stressed and unsolvable systems.

One candidate load flow solution method which has been shown to possess both speed and robustness for stressed and unsolvable systems is the optimal multiplier (OM) modification to the standard NR load flow. The OM load flow was first conceived in rectangular coordinates [9], but then extended using the same concepts to polar coordinates. Ref. [7] provides full details on how the method of [9] has been extended to polar coordinates with varying degrees of success. Although the OM load flow has been extended to polar coordinates, comparison to the equivalent formulation in rectangular coordinates has not been performed.

This is a crucial oversight, for the speed and robustness of a given load flow solution algorithm is dependent not only on the choice of algorithm but also on the choice of coordinate system used to represent the voltage phasors at the system buses [12]. Accordingly, this paper presents for the first time a direct comparison of the OM load flow with rectangular and polar coordinates using 2 bus, 118 bus, and 10,274 bus cases. For each case, the performance of the OM load flow is also compared to the standard NR load flow without optimal multipliers using each coordinate system. The rectangular OM load flow method used is that of Iwamoto and Tamura [9]. For the polar OM load flow, the method developed by Castro and Braz [6] was chosen due to its comparative advantages as demonstrated in [7].

These comparisons indicate that of the four methods considered—the OM load flow and NR load flow using polar and rectangular coordinates—the polar formulation of the OM load flow provides the best combination of speed and robustness for unstressed, stressed, and unsolvable systems.

II. The Rectangular and Polar OM Load Flow

A. The Load Flow Equations and the NR Algorithm [11]

In the polar NR load flow, the complex voltage phasor at each bus is represented using polar coordinates:

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$$\hat{V}_i = \left|\hat{V}_i\right| \angle \theta_i \tag{1}$$

For the rectangular load flow formulation, the voltage phasor at each bus is represented using rectangular coordinates:

$$\hat{V}_i = e_i + jf_i \tag{2}$$

In the NR load flow, the set of load flow equations $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ is solved. For the polar formulation, $\mathbf{f}(\mathbf{x})$ contains real and reactive power balance equations of the following forms:

$$\begin{aligned} P_i^{\text{Polar}}(\mathbf{x}) = \left|\hat{V}_i\right| \sum_{k \in C_i} \left\{ \left|\hat{V}_k\right| \left(G_{ik} \cos(\theta_i - \theta_k) + B_{ik} \sin(\theta_i - \theta_k) \right) \right\} \\ + P_{load,i} - P_{gen,i} = 0 \end{aligned} \tag{3}$$

$$\begin{aligned} Q_i^{\text{Polar}}(\mathbf{x}) = \left|\hat{V}_i\right| \sum_{k \in C_i} \left\{ \left|\hat{V}_k\right| \left(G_{ik} \sin(\theta_i - \theta_k) - B_{ik} \cos(\theta_i - \theta_k) \right) \right\} \\ + Q_{load,i} - Q_{gen,i} = 0 \end{aligned} \tag{4}$$

where $\hat{Y}_{ij} = G_{ij} + jB_{ij}$, the complex admittance between buses i and j, and C_i denotes the set of all buses connected to bus i, including itself. The most important aspect of these equations as they relate to the OM load flow is the presence of transcendental functions in (3) and (4).

For the rectangular formulation, $\mathbf{f}(\mathbf{x})$ contains real power balance, reactive power balance, and voltage setpoint equations of the following forms:

$$\begin{aligned} P_i^{\text{Rect}}(\mathbf{x}) = \sum_{k \in C_i} \left\{ e_i \left(G_{ik} e_k - B_{ik} f_k \right) + f_i \left(G_{ik} f_k + B_{ik} e_k \right) \right\} \\ + P_{load,i} - P_{gen,i} = 0 \end{aligned} \tag{5}$$

$$\begin{aligned} Q_i^{\text{Rect}}(\mathbf{x}) = \sum_{k \in C_i} \left\{ f_i \left(G_{ik} e_k - B_{ik} f_k \right) - e_i \left(G_{ik} f_k + B_{ik} e_k \right) \right\} \\ + Q_{load,i} - Q_{gen,i} = 0 \end{aligned} \tag{6}$$

$$V_i^{\text{Rect}}(\mathbf{x}) = e_i^2 + f_i^2 - \left|\hat{V}_i\right|_{specified}^2 = 0 \tag{7}$$

It should be noted that the rectangular formulation uses an extra equation at each PV bus in the system, due to the need to maintain the specified voltage magnitude (7). As a result, the rectangular formulation has a larger equation and variable count relative to the polar formulation, with the difference equal to the number of PV buses in the system.

The salient characteristic of these equations with regard to the OM load flow is that all the state variables in (5)-(7) appear in quadratic terms. This leads to significant simplification of these equations' Taylor series expansion, particularly with regard to the second-order term.

B. The OM Load Flow [6], [9]

The NR algorithm solves for the update vector $\Delta\mathbf{x}^{(\nu)}$ at each iteration ν based on the first-order Taylor series expansion of $\mathbf{f}(\mathbf{x})$, referred to hereafter as the linearization of $\mathbf{f}(\mathbf{x})$. The OM modification to the NR algorithm determines an optimal multiplier (i.e., an optimal step size) $\mu^{(\nu)}$ for the update vector based on the second-order Taylor expansion of each equation $f_i(\mathbf{x})$ in $\mathbf{f}(\mathbf{x})$ at iteration ν:

$$\begin{aligned} f_i\left(\mathbf{x}^{(\nu)} + \Delta\mathbf{x}^{(\nu)}\right) \approx \\ f_i\left(\mathbf{x}^{(\nu)}\right) + \left[\nabla_{\mathbf{x}} f_i\right]^{(\nu)} \left[\Delta\mathbf{x}^{(\nu)}\right] + \frac{1}{2}\left[\Delta\mathbf{x}^{(\nu)}\right]^T \left[\nabla_{\mathbf{x}}^2 f_i\right]^{(\nu)} \left[\Delta\mathbf{x}^{(\nu)}\right] \end{aligned} \tag{8}$$

An approximately equal sign is used in (8) to denote the inaccuracy of the second-order approximation for general equations. However, if the equation $f_i(\mathbf{x})$ is purely second-order (as in the rectangular form of the load flow equations), then (8) holds with strict equality. The optimal multiplier $\mu^{(\nu)}$ of the update vector $\Delta\mathbf{x}^{(\nu)}$ is determined by solving the following minimization problem:

$$\begin{aligned} F_i(\mu) &= \left\{ \begin{array}{l} f_i\left(\mathbf{x}^{(\nu)}\right) + \mu\left[\nabla_{\mathbf{x}} f_i\right]^{(\nu)}\left[\Delta\mathbf{x}^{(\nu)}\right] + \\ \frac{1}{2}\mu^2\left[\Delta\mathbf{x}^{(\nu)}\right]^T \left[\nabla_{\mathbf{x}}^2 f_i\right]^{(\nu)} \left[\Delta\mathbf{x}^{(\nu)}\right] \end{array} \right\} \\ \mathbf{F}(\mu) &= \left[F_1(\mu) \quad F_2(\mu) \quad \cdots \quad F_n(\mu) \right]^T \\ \mu^{(\nu)} &= \underset{\mu}{\operatorname{argmin}} \frac{1}{2}\left[\mathbf{F}(\mu)\right]^T \left[\mathbf{F}(\mu)\right] = \underset{\mu}{\operatorname{argmin}} \frac{1}{2}\left(\left\|\mathbf{F}(\mu)\right\|_2\right)^2 \end{aligned} \tag{9}$$

(9) can be easily solved as a cubic equation when optimality conditions are used [13]. In the case of multiple roots, the smallest real root is chosen as the optimal multiplier [9].

C. The Importance of Good Linearization of the Load Flow Equations

The optimal multiplier is only able to scale $\Delta\mathbf{x}^{(\nu)}$; the direction of the update vector is still based entirely on the first-order Taylor series expansion as in the NR algorithm. Accordingly, if the linearization of $\mathbf{f}(\mathbf{x})$ is poor, $\Delta\mathbf{x}^{(\nu)}$ may not indicate a very good direction. When the direction is not very good, the optimal multiplier provides little help in solving the system and can even slow down the solution.

This behavior can be seen mathematically by examining the effect of large second-order terms (i.e., poor linearization) on the optimal multiplier. The only information used when computing the optimal multiplier $\mu^{(\nu)}$ that is not used in computing the update vector $\Delta\mathbf{x}^{(\nu)}$ is the second-order term (denoted as $\mathbf{c}^{(\nu)}$) from the Taylor expansion of $\mathbf{f}(\mathbf{x})$:

$$\begin{aligned} c_i^{(\nu)} &= \frac{1}{2}\left[\Delta\mathbf{x}^{(\nu)}\right]^T \left[\nabla_{\mathbf{x}}^2 f_i\right]^{(\nu)} \left[\Delta\mathbf{x}^{(\nu)}\right] \\ &= \sum_{k=1}^{n} \sum_{m=1}^{n} \left. \frac{\partial^2 f_i(\mathbf{x})}{\partial x_m \partial x_k} \right|_{\mathbf{x}=\mathbf{x}^{(\nu)}} \Delta x_m \Delta x_k \\ \mathbf{c}^{(\nu)} &= \left[c_1^{(\nu)} \quad c_2^{(\nu)} \quad \cdots \quad c_n^{(\nu)} \right]^T \end{aligned} \tag{10}$$

The notation $\mathbf{c}^{(\nu)}$ is used to be consistent with the historical OM formulation [9]. Because $\mathbf{c}^{(\nu)}$ is the second-order term of the Taylor expansion, $\mathbf{c}^{(\nu)}$ is zero if the load flow equations are exactly equal to their first-order Taylor series expansion. That is,

$$\mathbf{f}\left(\mathbf{x}^{(\nu)}+\Delta\mathbf{x}^{(\nu)}\right)=\mathbf{f}\left(\mathbf{x}^{(\nu)}\right)+\left[\nabla_{\mathbf{x}}\mathbf{f}\right]^{(\nu)}\left[\Delta\mathbf{x}^{(\nu)}\right] \tag{11}$$

When the linearization of $\mathbf{f}\left(\mathbf{x}^{(\nu)}+\Delta\mathbf{x}^{(\nu)}\right)$ is not exact, the equality in (11) does not hold and $\mathbf{c}^{(\nu)}$ can take on a wide range of values.

If $\mathbf{c}^{(\nu)}$ is zero, the optimal multiplier will be 1, just as if the NR algorithm were used. On the other hand, if $\mathbf{c}^{(\nu)}$ is much larger than the other terms in (8), (9) can be reduced to:

$$\mu^{(\nu)} \approx \underset{\mu}{\operatorname{argmin}} \frac{1}{2}\mu^2\left(\left\|\mathbf{c}^{(\nu)}\right\|_2\right)^2 \tag{12}$$

The solution of (12) is $\mu^{(\nu)} \approx 0$, indicating that a large second-order term in the Taylor expansion leads to a very small optimal multiplier value.

A small optimal multiplier is desirable when attempting to solve unsolvable systems; it is precisely because $\mu^{(\nu)}$ takes on small values in such cases that divergence of the OM algorithm is prevented. On the other hand, if small optimal multipliers show up when solving solvable systems, then the algorithm can take more iterations than the standard NR algorithm (which can be thought of as using a constant multiplier of 1).

III. Advantages and Disadvantages of Each Coordinate System for the OM Load Flow

A. Advantages and Disadvantages of Using Polar Coordinates with the OM Load Flow

The polar formulation has several advantages over the rectangular formulation when solving the load flow using the OM solution method. The polar form of the load flow equations exhibit excellent linearization characteristics, as demonstrated by both the Fast Decoupled Load Flow (FDLF) [14] and the use of sensitivity factors in power system analysis. Also, the polar formulation of the load flow equations has fewer equations to solve than the rectangular formulation. This can be significant for systems with relatively high percentages of PV buses as in the IEEE 118 bus system.

However, there are also disadvantages to using the polar formulation instead of the rectangular formulation. The most significant drawback to the polar formulation is the presence of transcendental functions. These functions lead to infinite order terms in the Taylor expansion, which makes (8) an approximation rather than a strict equality as in the rectangular formulation. As a result, the calculation of $\mu^{(\nu)}$ can be less accurate than with the rectangular formulation which does not have any Taylor series expansion terms of order higher than two. Also, the presence of sine and cosine in the polar load flow equations results in a more complex calculation of the second-order term $\mathbf{c}^{(\nu)}$ needed to solve for the optimal multiplier. Fortunately, the calculation of $\mathbf{c}^{(\nu)}$ is still on the order of a mismatch calculation [6], [7]. Though there are some disadvantages to using the polar formulation, the case studies will demonstrate that these disadvantages are outweighed by the advantages of using the polar formulation of the load flow equations and variables with the OM load flow.

B. Advantages and Disadvantages of Using Rectangular Coordinates with the OM Load Flow

In the original derivation [9] of the OM solution method, several key advantages of the rectangular formulation are given. The greatest benefit of using the rectangular formulation results from the quadratic nature of the load flow equations when rectangular coordinates are used. Because all the state variables appear in quadratic terms in the equations, the third and higher order terms of the Taylor expansion are zero; this makes (8) hold with strong equality. This can lead to greater accuracy in the calculation of $\mu^{(\nu)}$ relative to the polar formulation. Further, $\mathbf{c}^{(\nu)}$ is much easier to calculate and program than with the polar formulation, because in the rectangular formulation:

$$\mathbf{c}^{(\nu)}=\mathbf{f}\left(\Delta\mathbf{x}^{(\nu)}\right) \tag{13}$$

As a result, calculating the second-order term is just a matter of plugging $\Delta\mathbf{x}^{(\nu)}$ into the mismatch calculation routines instead of $\mathbf{x}^{(\nu)}$.

Unfortunately, there are also several disadvantages to using the rectangular formulation. One problem with the rectangular formulation is the lack of widespread implementation of the NR load flow in rectangular coordinates [7], though at least one commercial load flow package does use the rectangular formulation by default [15]. The extremely poor performance of the decoupled load flow in rectangular coordinates [16] also indicates that the rectangular formulation may not have as good of a linearization as the polar formulation. The extra voltage mismatch equation that must be satisfied at PV buses (7) can also lead to difficulties with the OM algorithm in rectangular coordinates [17]. The two bus PV system examined below in the case studies clearly demonstrates how the extra equation can cause trouble with the OM algorithm.

IV. Case Studies

Because the convergence behavior of the NR load flow is difficult to analyze mathematically, particularly when the OM modification is used, empirical results are used to compare the two formulations.

A solution tolerance of 0.01 MW was used for each simulation. As discussed in [7] and [9], the reduction of the optimal multiplier to very small values indicates unsolvability of the associated power system. For the following cases, the solution was stopped when the optimal multiplier dropped below 0.01, indicating that the solution had stalled at a constant mismatch and the system was unsolvable. It should also be noted that all four methods indicated that the same sets of cases were unsolvable, giving further credence to the notion that small optimal multipliers do indeed indicate unsolvability of the underlying power system. If any islanding occurred during outage studies, results were taken from the largest island.

In the following sections, ROM (POM) refers to solution using the OM load flow with rectangular (polar) coordinates, and RNR (PNR) refers to solution using the standard NR load flow without optimal multipliers with rectangular (polar) coordinates. In reporting the results for the case studies, several indices are used:

$$IC_{Rect}^{no\ opt}\left(IC_{Polar}^{no\ opt}\right) = \text{\# of iterations to solve with RNR (PNR)} \tag{14}$$

$$IC_{Rect}^{opt}\left(IC_{Polar}^{opt}\right) = \text{\# of iterations to solve with ROM (POM)} \tag{15}$$

$$\Gamma_{opt} = IC_{Rect}^{opt} - IC_{Polar}^{opt} \tag{16}$$

$$\Gamma_{no\ opt} = IC_{Rect}^{no\ opt} - IC_{Polar}^{no\ opt} \tag{17}$$

$$\Delta_{Rect} = IC_{Rect}^{no\ opt} - IC_{Rect}^{opt} \tag{18}$$

$$\Delta_{Polar} = IC_{Polar}^{no\ opt} - IC_{Polar}^{opt} \tag{19}$$

Values of Γ_{opt} ($\Gamma_{no\ opt}$) greater than zero indicate poorer performance of ROM (RNR) relative to POM (PNR). Values of Δ_{Rect} (Δ_{Polar}) greater than zero indicate poorer performance of ROM (POM) relative to RNR (PNR).

To examine the performance of the OM and NR solution methods under both coordinate systems, four systems are examined—a two bus system with a PQ bus, a two bus system with a PV bus, the IEEE 118 bus system, and a 10,274 bus system.

A. Two Bus PQ System

1) System Description

First, a simple two bus system is examined in detail to demonstrate the behavior of the four solution methods. The system has a slack bus (bus 1) and an unregulated (PQ) load bus (bus 2) connected by a line with X held constant at 0.01 p.u. In all cases, initial conditions were taken to be the flat start values.

2) Case Studies

Cases were generated for this system by simultaneously varying three parameters: MW load from 0 to 2500 MW, Mvar load from 0 to 2500 Mvar, and R/X ratio from 0 to 2. Each range of system parameters (MW, Mvar, and R) was broken up into 30 points, giving 27,000 total cases. 13,209 of the cases were solvable, while the remaining 13,791 were not solvable. A comparison of the number of iterations required for the various methods is presented in Table I and Table II.

POM provides significant gains over PNR, ROM, and RNR for these cases. Compared to PNR, POM takes an average of 51% fewer iterations for solvable cases, indicating significant performance gains when using the OM algorithm instead of the NR algorithm for these cases. RNR and ROM also performed worse than POM, taking an average of 52% and 27% more iterations, respectively.

B. Two Bus PV System

1) System Description

Another two bus system is examined to look at the effects of the voltage setpoint equation in rectangular coordinates as in [17]. For this case, bus 2 has a voltage regulating generator in addition to a load, making bus 2 a PV bus instead of a PQ bus. The line connecting the two buses has constant parameters of R = 0.005 p.u. and X = 0.01 p.u. for all cases. Because the voltage at bus 2 is regulated, there is only one equation ($P_2^{\text{Polar}}\left(\theta_2\right)$ = 0) and one unknown (θ_2) for this system in polar coordinates. However, though the polar solution is trivial, this system clearly demonstrates the problems that can arise with ROM due to the voltage setpoint mismatch equation.

TABLE I – NUMBER OF ITERATIONS FOR TWO BUS (PQ) SOLVABLE CASES

	Min	Max	Avg
IC_{Polar}^{opt}	1	6	2.56
IC_{Rect}^{opt}	1	7	3.25
$IC_{Polar}^{no\ opt}$	1	9	3.87
$IC_{Rect}^{no\ opt}$	1	9	3.90

	% of Cases		
	> 0	= 0	< 0
Γ_{opt}	66.76%	32.48%	0.76%
$\Gamma_{no\ opt}$	3.29%	96.28%	0.44%
Δ_{Rect}	0.00%	41.38%	58.62%
Δ_{Polar}	0.00%	11.57%	88.43%

TABLE II – NUMBER OF ITERATIONS FOR TWO BUS (PQ) UNSOLVABLE CASES

	Min	Max	Avg
IC_{Polar}^{opt}	2	6	2.47
IC_{Rect}^{opt}	2	9	3.00

	% of Cases		
	> 0	= 0	< 0
Γ_{opt}	45.34%	54.14%	0.51%

2) Case Studies

In these cases the MW load at the second bus was varied from 0 to 4944 MW (maximum loadability) in increments of 1 MW, resulting in 4944 solvable cases. Due to the trivial solution of this system with polar coordinates, only the results for ROM and RNR are given in Table III.

TABLE III – NUMBER OF ITERATIONS FOR TWO BUS (PV) CASES

	Min	Max	Avg
IC_{Rect}^{opt}	0	11	5.22
$IC_{Rect}^{no\ opt}$	0	9	3.71

	% of Cases		
	> 0	= 0	< 0
Δ_{Rect}	70.89%	27.57%	1.54%

Fig. 1 shows the relationship between loading level and Δ_{Rect} for this case. The maximum value of Δ_{Rect} for the load range shown is 7, and the minimum value is -3. While Δ_{Rect} varied widely for these cases, the ROM took more iterations in 71% of the cases. While the ROM performed poorly for the vast majority of unstressed cases, as the system approached unsolvability the ROM did see some performance gains over RNR.

An unusual feature of Fig. 1 is the large increase in Δ_{Rect} for load levels around 4000 MW (about 80% of the maximum). A more detailed look at the system mismatch equations for this load level can help to explain why. Fig. 2 and Fig. 3 are plots of the absolute value of the voltage setpoint mismatch equation (7) and the real power mismatch equation (5) at bus 2 as a function of the real and imaginary components of the bus 2 voltage over the ranges $0.75 \le e_2 \le 1.00$ and $-0.95 \le f_2 \le -0.4$. Of course the goal of the NR algorithm is to determine the point where both mismatch equations are equal to zero. For a load of 4000 MW this occurs at V_2 = 0.8 - j0.6. Clearly, evaluation of (5) results in values several orders of magnitude higher than (7) in the region of interest. As a result, the real power mismatch dominates both the shape and magnitude of the 2-norm of the total mismatch of the system, which is shown in Fig. 4.

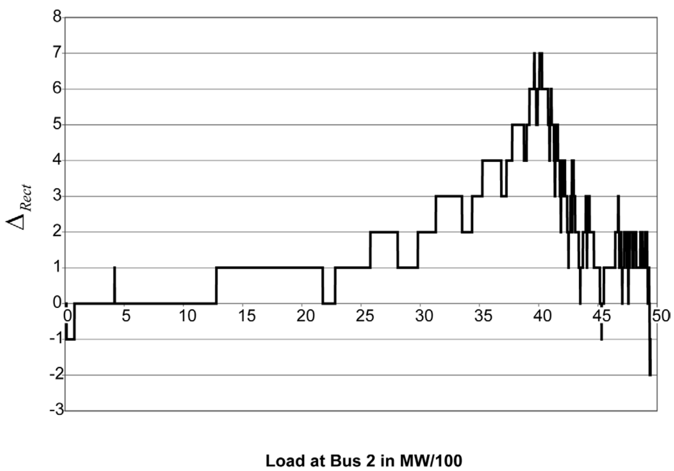

Fig. 1. The relationship between Δ_{Rect} and the MW load at bus 2 for the two bus PV system

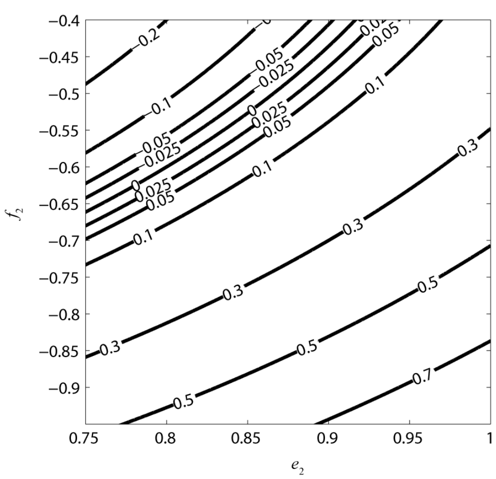

Fig. 2. Voltage mismatch (7) for two bus PV case with 4000 MW load

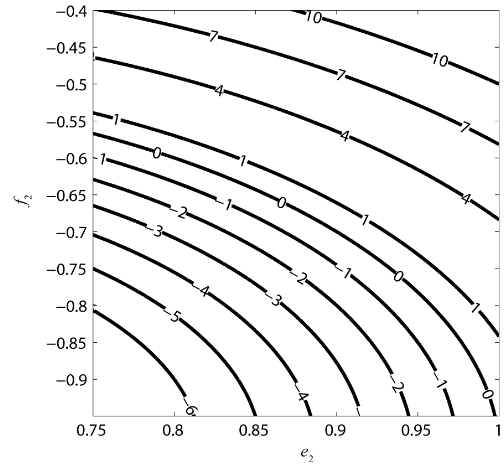

Fig. 3. Real power mismatch (5) for two bus PV case with 4000 MW load

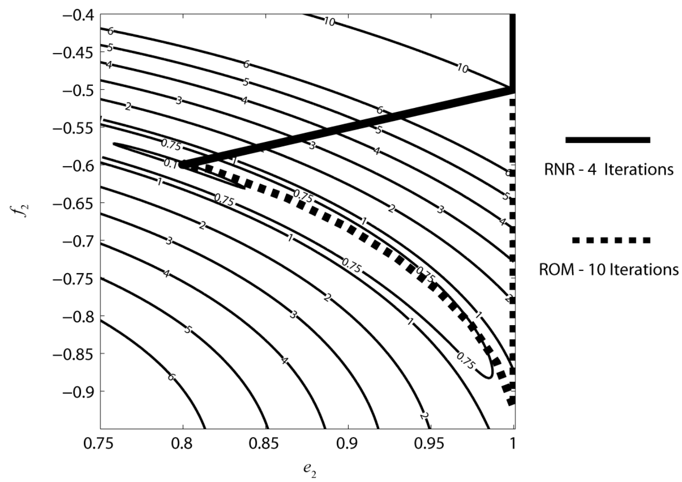

Fig. 4. Total mismatch and iteration paths for the two bus PV case (4000 MW load) using ROM and RNR

Also plotted in Fig. 4 are the solution paths taken with RNR (solid line) and ROM (dashed line). Both iterations begin at a flat start of e_2 = 1.0, f_2 = 0.0 and end with e_2 = 0.8, f_2 = -0.6. Solving with the ROM clearly takes a longer path than solving with RNR. The ROM solution method first sets the real power mismatch to zero then attempts to correct the voltage mismatch while keeping the real power mismatch very close to zero.

The cause of this behavior is precisely the magnitude difference mentioned above. Because the real power mismatch overpowers the voltage mismatch in the 2-norm, the ROM solution method forces the solution to always stay very close to the region where the real power mismatch is zero. This requires the solution to crawl along the $\left|P_2^{\mathrm{Rect}}(\mathbf{x})\right|$ = 0 curve. In this case, because the first iteration puts the voltage values at bus 2 far from the correct values for the voltage setpoint equation, the solution must wind along the $\left|P_2^{\mathrm{Rect}}(\mathbf{x})\right|$ = 0 curve to get to the final solution. The tight, slow traversal of the $\left|P_2^{\mathrm{Rect}}(\mathbf{x})\right|$ = 0 curve to arrive at the final solution is responsible for the difference in iteration counts between ROR and RNR.

In summary, the voltage mismatch equation does not present much of a challenge for the traditional NR load flow (RNR); convergence proceeds normally. On the other hand, the Newton-Raphson load flow with the usage of optimal multipliers (ROM) can encounter significant problems due to the vast differences of scale caused by the voltage setpoint equation at PV buses.

Heuristic methods of alleviating this problem, e.g. scaling the voltage equation by a fixed magnitude and disallowing small optimal multipliers, are discussed in [15] and [17]. Unfortunately, both of these methods have their own pitfalls. Scaling the voltage equation is problematic due to the difficulty in determining exactly how much to scale each voltage setpoint equation in large systems, and the rejection of small optimal multipliers can have the undesired side effect of causing more iterations to be performed for unsolvable systems.

C. IEEE 118 Bus System

1) System Description

The IEEE 118 bus system [18] is examined next. In order to compare the performance of the rectangular and polar formulations with this system, three difference studies were performed—all single outages, all double outages, and system-wide load scaling. Flat start values of $1\angle 0°$ p.u. were used as initial conditions for each case.

2) Single and Double Outage Studies

For the single outage study, all 186 lines in the system were outaged and the solution results were compared. The system was solvable for all single outages. For the double outage study, all 186 lines in the system were outaged in pairs for a total of 17,205 cases. One double outage case was unsolvable; in that case, the rectangular formulation took 5 iterations to stall and the polar formulation took 4. A comparison of the number of iterations required is given in Table IV for all 17,390 solvable outage cases.

TABLE IV – NUMBER OF ITERATIONS FOR 118 BUS SINGLE AND DOUBLE OUTAGE CASES

	Min	Max	Avg
IC^{opt}_{Polar}	3	4	3.001
IC^{opt}_{Rect}	3	6	3.166
$IC^{no\ opt}_{Polar}$	3	4	3.022
$IC^{no\ opt}_{Rect}$	3	6	4.012

	% of Cases		
	> 0	= 0	< 0
Γ_{opt}	16.44%	83.56%	0.00%
$\Gamma_{no\ opt}$	98.86%	1.14%	0.00%
Δ_{Rect}	0.05%	15.34%	84.61%
Δ_{Polar}	0.00%	97.81%	2.19%

Though the polar formulation did not see much improvement with the usage of optimal multipliers for the outage cases, IC^{opt}_{Polar} still has the lowest average of the four solution methods. The most notable aspect of these results, however, is that in 98.86% of the outage cases studied, PNR performed better than RNR. This is most likely due to the high number of PV buses in this case—47 out of the 118 buses.

3) Load Scaling

For the load scaling study, all real and reactive loads and generator outputs in the system were scaled uniformly by a multiplier. This multiplier ranged from 0.001 to 4.000, and was incremented by 0.001 for each case, giving a total of 4000 cases. The system was solvable for scaling between 0.001 and 3.187, and was unsolvable for scaling between 3.188 and 4.000. Comparison of the number of iterations required for these cases is summarized in Table V and Table VI.

Several aspects of the results for the load scaling are quite interesting. Most importantly, the average iteration count for POM is well below the other three methods, mirroring the results seen for the two bus PQ cases. Also, in all of the 813 unsolvable cases, POM stalled in fewer iterations than ROM. Because one of the primary purposes of using optimal multipliers with the Newton-Raphson algorithm is to quickly stall at a constant mismatch for unsolvable cases, the performance of the rectangular formulation for these unsolvable cases is of great concern.

ROM also performed worse relative to POM as the load multiplier was increased. In Fig. 5, the solid line represents solvable cases and the dashed line represents unsolvable cases. Due to the large power transfers needed to satisfy the scaled load demand, large angle changes are occurring along with the load scaling; the norm of all angle changes on the system has a correlation coefficient of 0.91 with the load scaling. This suggests that large angle shifts are tied to the poor performance of ROM.

TABLE V – NUMBER OF ITERATIONS FOR THE IEEE 118 BUS LOAD SCALING CASES (SOLVABLE)

	Min	Max	Avg
IC^{opt}_{Polar}	3	7	3.55
IC^{opt}_{Rect}	3	13	4.53
$IC^{no\ opt}_{Polar}$	3	9	3.70
$IC^{no\ opt}_{Rect}$	3	10	4.29

	% of Cases		
	> 0	= 0	< 0
Γ_{opt}	63.85%	36.15%	0.00%
$\Gamma_{no\ opt}$	57.92%	42.08%	0.00%
Δ_{Rect}	26.26%	66.74%	7.00%
Δ_{Polar}	0.00%	85.00%	15.00%

TABLE VI – NUMBER OF ITERATIONS FOR THE 118 BUS LOAD SCALING CASES (UNSOLVABLE)

	Min	Max	Avg
IC^{opt}_{Polar}	4	7	4.58
IC^{opt}_{Rect}	10	14	12.45

	% of Cases		
	> 0	= 0	< 0
Γ_{opt}	100%	0%	0%

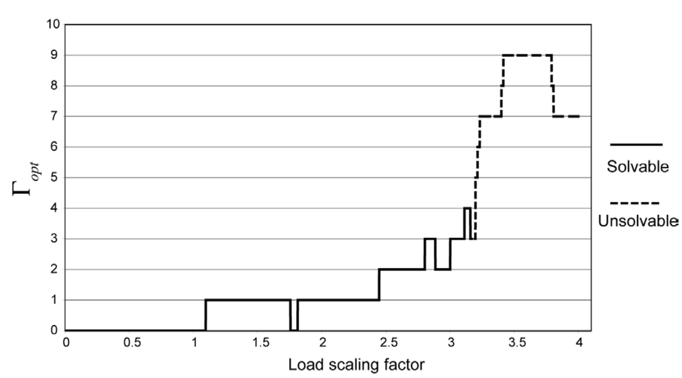

Fig. 5. Relation between the load scaling for the 118 bus system and Γ_{opt}

D. 10,274 Bus System

1) System Description

The final system examined is a 10,274 bus case. To test this case, the 500 lines carrying the most power were outaged and the solutions were examined. Of the 500 outage cases studied, 479 were solvable and 21 unsolvable. A comparison of the number of iterations for the solvable and unsolvable cases are provided in Table VII and Table VIII, respectively.

These results are simply terrible for the rectangular formulation. In 33 of the 479 solvable outage cases the rectangular formulation took at least 20 iterations to solve with the optimal multiplier. This accounts for the very large average value of IC_{Rect}^{opt} in Table VII . For the 21 unsolvable cases, Γ_{opt} was greater than zero in all cases. The rectangular formulation performed extraordinarily poorly for the unsolvable 10,274 bus cases. For instance, in 3 of the unsolvable cases, it took over 100 iterations for the rectangular optimal multiplier to drop below 0.01. As in the other systems, IC_{Polar}^{opt} has the lowest average iteration count of the four methods used to solve the load flow.

As in the 118 bus load scaling cases, there are some clear dependencies between angle shifts and problems with ROM in this system. Fig. 6 provides a visual indicator of this dependence. Each dot in this figure represents a single solvable outage case. The correlation coefficient between the norm of the angle shifts and the differences in iteration counts is 0.98, based on the 479 solvable outage cases.

TABLE VII - NUMBER OF ITERATIONS FOR THE 10,274 BUS OUTAGE CASES (SOLVABLE)

	Min	Max	Avg
IC_{Polar}^{opt}	2	5	2.61
IC_{Rect}^{opt}	3	85	6.06
$IC_{Polar}^{no\ opt}$	2	6	2.82
$IC_{Rect}^{no\ opt}$	3	7	3.38

	% of Cases		
	> 0	= 0	< 0
Γ_{opt}	70.15%	29.85%	0.00%
$\Gamma_{no\ opt}$	50.94%	48.85%	0.21%
Δ_{Rect}	16.28%	81.21%	2.51%
Δ_{Polar}	0.00%	79.12%	20.88%

TABLE VIII - NUMBER OF ITERATIONS FOR THE 10,274 BUS OUTAGE CASES (UNSOLVABLE)

	Min	Max	Avg
IC_{Polar}^{opt}	3	8	5.10
IC_{Rect}^{opt}	11	144	68.33

	% of Cases		
	> 0	= 0	< 0
Γ_{opt}	100%	0%	0%

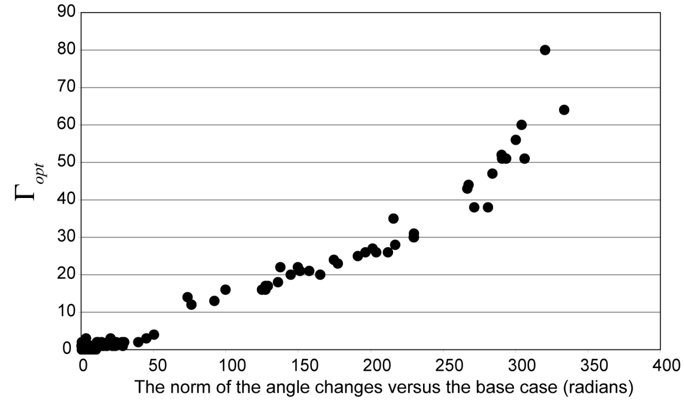

Fig. 6. Relation between the 2-norm of the angle shifts for the 10,274 bus system and Γ_{opt}

V. DISCUSSION OF THE CASE STUDIES

A. The Effect of Angle Shifts

For the mid- to large-sized stressed systems examined in this paper (the IEEE 118 bus load scaling cases and the 10,274 bus outage cases), the greatest single indicator of poor performance with the rectangular formulation is the norm of the angle shifts for the system. The most likely cause of this dependence is that a change in angle is a curve in the rectangular solution space rather than a straight line. Also, as shown for the two bus case in Fig. 4, the rectangular formulation can have great difficulty in moving along a curve when the optimal multiplier is employed. For the polar formulation, on the other hand, changing angles is a linear movement with respect to the solution variables. This could help to explain why the polar formulation does not exhibit the same performance degradation when large angle shifts occur.

B. Average Iteration Count Differences

For all three system sizes, the average value of IC_{Polar}^{opt} is less than the average value of IC_{Rect}^{opt}, indicating that the polar formulation usually performs better than the rectangular formulation when optimal multipliers are used. Also, in each set of cases, the average value of $IC_{Polar}^{no\ opt}$ is less than the average value of $IC_{Rect}^{no\ opt}$. This shows that the polar formulation routinely performs better than the rectangular formulation whether or not optimal multipliers are not used. Finally, the average value of $IC_{Polar}^{no\ opt}$ is greater than the average value of IC_{Polar}^{opt} for all three system sizes, indicating that the polar usually receives some benefit from the usage of the optimal multiplier for solvable cases. On the other hand, the rectangular formulation does worse when optimal multipliers are used for several of the 10,274 bus cases and for the majority of the two bus PV cases.

VI. CONCLUSIONS

The case studies indicate that any advantages of using the rectangular formulation are offset by greater difficulties. These problems are particularly apparent as the system becomes highly stressed and unsolvable. At its best (the two bus PQ system), ROM took an average of 0.5 iterations more than POM in stalling for unsolvable cases. In the worst case, the 10,274 bus case, ROM took an average of 63 iterations more than POM to stall, including quite a few cases which took unreasonably long times (over 100 iterations) to stall.

On the other hand, the polar form of the OM algorithm performed very well throughout all simulations. The POM had a lower average iteration count than ROM for all systems, indicating that the polar coordinate system is the best choice if the OM algorithm is to be used. Also, POM behaved well for unsolvable systems by stalling quickly and not diverging. This behavior gives a clear advantage over both RNR and PNR when dealing with unsolvable systems, as the standard NR algorithm does not handle unsolvable systems gracefully. POM also had a lower average iteration count than PNR for every set of cases (and never had a higher iteration count than PNR for any single case). For these reasons, the authors recommend implementation of the optimal multiplier modification to the Newton-Raphson load flow with polar coordinates to get the fastest, most robust performance, regardless of system solvability or size.

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