ECEN 667 HW1 SOLUTIONS.

Answer 1

$\Delta t = 0.1$

Initial values: $x_1(0) = 1$

$x_2(0) = 1$

find values of $x_1(0.4)$ and $x_2(0.4)$

$$\dot{x}_1 = \frac{2}{3}x_1 - \frac{5}{3}x_1x_2 \qquad \longrightarrow \qquad \frac{dx_1}{dt} = \frac{2}{3}x_1(t) - \frac{5}{3}x_1(t)x_2(t)$$

$$\dot{x}_2 = x_1x_2 - x_2 \qquad \qquad \frac{dx_2}{dt} = x_1(t)x_2(t) - \frac{1}{2}x_2(t)$$

$$\rightarrow f(x(t)) = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}x_1(t) - \frac{5}{3}x_1(t)x_2(t) \\ x_1(t)\cdot x_2(t) - \frac{1}{2}x_2(t) \end{bmatrix}$$

Problem setup for Second Order RK method:

$$x(t+\Delta t) = x(t) + \frac{1}{2}(k_1 + k_2)$$

where;

$$k_1 = \Delta t f(x)$$

$$k_2 = \Delta t f(x + k_1)$$

Iteration I at $t=0$.

$$k_1 = \Delta t \cdot f(x) = (0.1)\begin{bmatrix} \frac{2}{3}x_1(0) - \frac{5}{3}x_1(0)x_2(0) \\ x_1(0)\cdot x_2(0) - x_2(0) \end{bmatrix}$$

$$= (0.1)\begin{bmatrix} \frac{2}{3} - \frac{5}{3} \\ 1 - 1 \end{bmatrix} = 0.1\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.1 \\ 0 \end{bmatrix} = k_1$$

$$k_2 = \Delta t f(x+k_1) = 0.1 f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{pmatrix} -0.1 \\ 0 \end{pmatrix}\right) = 0.1 f\left(\begin{bmatrix} 0.9 \\ 1 \end{bmatrix}\right)$$

$$= 0.1\begin{bmatrix} \frac{2}{3}(0.9) - \frac{5}{3}(0.9)(1.0) \\ (0.9)(1.0) - (1.0) \end{bmatrix} = 0.1\begin{bmatrix} -0.9 \\ -0.1 \end{bmatrix} = \begin{bmatrix} -0.09 \\ -0.01 \end{bmatrix} = k_2$$

$$x(0+0.1) = x(0) + \frac{1}{2}(k_1+k_2)$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\left(\begin{bmatrix} -0.1 \\ 0 \end{bmatrix} + \begin{bmatrix} -0.09 \\ -0.01 \end{bmatrix}\right) = \begin{bmatrix} 0.905 \\ 0.995 \end{bmatrix} = x(0.1)$$

Iteration 2 at $t=0.1$

$$k_1 = \Delta t f(x) = 0.1 \times \begin{bmatrix} \frac{2}{3}(0.905) - \frac{5}{3}(0.905)(0.995) \\ (0.905)(0.995) - (0.905) \end{bmatrix} = \begin{bmatrix} -0.0897 \\ -0.0095 \end{bmatrix} = k_1$$

$$k_2 = \Delta t f(x+k_1) = 0.1 \times f\left(\begin{bmatrix} 0.905 \\ 0.995 \end{bmatrix} + \begin{bmatrix} -0.0897 \\ -0.0095 \end{bmatrix}\right) = 0.1 \times f\left(\begin{bmatrix} 0.8153 \\ 0.985 \end{bmatrix}\right) = \begin{bmatrix} -0.0795 \\ -0.0182 \end{bmatrix}$$

$$x(0.1+0.1) = x(0.1) + \frac{1}{2}(k_1+k_2)$$

$$= \begin{bmatrix} 0.905 \\ 0.995 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -0.169 \\ -0.027 \end{bmatrix} = \begin{bmatrix} 0.8205 \\ 0.9815 \end{bmatrix} = x(0.2)$$

Iteration III at $t = 0.2$

$$k_1 = 0.1 \times \begin{bmatrix} \frac{2}{3}(0.8205) - \frac{5}{3}(0.8205)(0.9815) \\ (0.8205)(0.9815) - (0.9815) \end{bmatrix} = \begin{bmatrix} -0.0795 \\ -0.0176 \end{bmatrix}$$

$$k_2 = 0.1\, f(x(0.2) + k_1) = 0.1 \times f\left(\begin{bmatrix} 0.8205 \\ 0.9815 \end{bmatrix} + \begin{bmatrix} -0.0795 \\ -0.0176 \end{bmatrix}\right) = 0.1 \times f\left(\begin{bmatrix} 0.741 \\ 0.964 \end{bmatrix}\right) = \begin{bmatrix} -0.069 \\ -0.025 \end{bmatrix}$$

$$x(0.2+0.1) = x(0.2) + \frac{1}{2}(k_1 + k_2)$$

$$= \begin{bmatrix} 0.8205 \\ 0.9815 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -0.1485 \\ -0.0426 \end{bmatrix} = \begin{bmatrix} 0.7463 \\ 0.9602 \end{bmatrix} = x(0.3)$$

Iteration IV at $t = 0.3$

$$k_1 = 0.1 \begin{bmatrix} \frac{2}{3}(0.7463) - \frac{5}{3}(0.9602)(0.7463) \\ (0.7463)(0.9602) - (0.9602) \end{bmatrix} = \begin{bmatrix} -0.0697 \\ -0.0244 \end{bmatrix}$$

$$k_2 = 0.1\, f(x(0.3) + k_1) = 0.1 \times f\left(\begin{bmatrix} 0.6766 \\ 0.9358 \end{bmatrix}\right) = \begin{bmatrix} -0.0604 \\ -0.0302 \end{bmatrix}$$

$$x(0.3+0.1) = x(0.4) = \begin{bmatrix} 0.7463 \\ 0.9602 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -0.06505 \\ -0.0546 \end{bmatrix} = \begin{bmatrix} 0.6813 \\ 0.9329 \end{bmatrix} = x(0.4)$$

t	$x_1(t)$	$x_2(t)$
0.1	0.905	0.995
0.2	0.8205	0.9815
0.3	0.7463	0.9602
0.4	0.6813	0.9329

Answer 2

Using $f(x(0))$ and $f(x(0.1))$ from answer to Ques 1.,

$$f(x(0.1)) = \begin{bmatrix} 0.897 \\ 0.095 \end{bmatrix} \quad \& \quad f(x(0)) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

at $t = 0.1$ $(\Delta t = 0.1)$

$$x(0.1+0.1) = x(0.1) + \frac{0.1}{2}\left[3 \cdot f(x(0.1)) - f(x(0)) + O(\Delta t^3)\right]$$

$$x(0.2) = \begin{bmatrix} 0.905 \\ 0.995 \end{bmatrix} + \frac{0.1}{2}\left[3 \cdot \begin{bmatrix} -0.897 \\ -0.095 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 0.8204 \\ 0.9807 \end{bmatrix} \longrightarrow f(x(0.2)) = \begin{bmatrix} -0.7941 \\ -0.1761 \end{bmatrix}$$

at $t = 0.2$

$$x(0.2+0.1) = x(0.2) + \frac{0.1}{2}\left[3 \cdot f(x(0.2)) - f(x(0.1))\right]$$

$$= \begin{bmatrix} 0.8204 \\ 0.9807 \end{bmatrix} + \frac{0.1}{2}\left[3 \cdot \begin{bmatrix} -0.7941 \\ -0.1761 \end{bmatrix} - \begin{bmatrix} -0.897 \\ -0.095 \end{bmatrix}\right] = \begin{bmatrix} 0.7461 \\ 0.959 \end{bmatrix} \longrightarrow f(x(0.3)) = \begin{bmatrix} -0.695 \\ -0.244 \end{bmatrix}$$

at $t = 0.3$

$$x(0.4) = x(0.3) + \frac{0.1}{2}\left[3\cdot f(x(0.3)) - f(x(0.2))\right]$$
$$= \begin{bmatrix} 0.7461 \\ 0.959 \end{bmatrix} + \frac{0.1}{2}\left[3\cdot\begin{bmatrix} -0.695 \\ -0.244 \end{bmatrix} - \begin{bmatrix} -0.7941 \\ -0.1761 \end{bmatrix}\right]$$
$$= \begin{bmatrix} 0.6815 \\ 0.9312 \end{bmatrix}$$
$$\Rightarrow f(x(0.4)) = \begin{bmatrix} -0.6034 \\ -0.296 \end{bmatrix}$$

t	$x_1(t)$	$x_2(t)$
0.1	0.905	0.995
0.2	0.8204	0.9807
0.3	0.7461	0.959
0.4	0.6815	0.9312

Answer 3

$$\dot{x}_1 = \frac{2}{3}x_1 - \frac{5}{3}x_1\cdot x_2$$
$$\dot{x}_2 = x_1\cdot x_2 - x_2$$

At equilibrium, $\dot{x}_1 = \dot{x}_2 = 0$

$$\frac{2}{3}x_1 - \frac{5}{3}x_1x_2 = 0$$
$$x_1\cdot\left(\frac{2}{3} - \frac{5}{3}x_2\right) = 0$$
$$x_1 = 0, \quad x_2 = \frac{2}{5}$$

$$x_1x_2 - x_2 = 0$$
$$(x_1 - 1)\cdot x_2 = 0$$
$$x_2 = 0, \quad x_1 = 1$$

$\therefore$ Two equilibrium point are $(0,0)$ and $\left(1, \frac{2}{5}\right)$.

Answer 4.

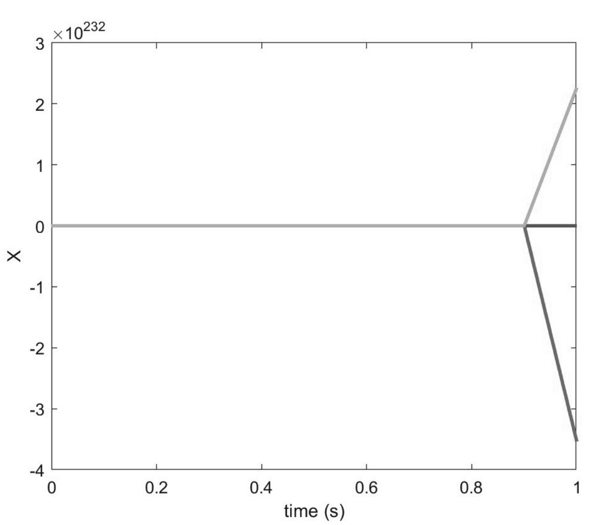
$$\dot{x}_1 = 8(x_2 - x_1)$$
$$\dot{x}_2 = x_1(28 - x_3) - x_2$$
$$\dot{x}_3 = x_1x_2 - \frac{4}{3}x_3.$$

initial conditions,

$$x_1(0) = x_2(0) = x_3(0) = 5$$

$$f(x(t)) = \begin{bmatrix} 8(x_2 - x_1) \\ x_1(28 - x_3) - x_2 \\ x_1x_2 - \frac{4}{3}x_3 \end{bmatrix}$$

Using matlab, the following plot is obtained for 10 time steps;



→ this shows that the function does not converge to an equilibrium point!

Equilibrium points occur when $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0$

$8(x_2 - x_1) = 0$

$\Rightarrow x_2 = x_1$ — (1)

$x_1(28 - x_3) - x_2 = 0$

$28x_1 - x_1x_3 = x_2$

Using eqn (1),

$28x_1 - x_1x_3 = x_1$

$27 = x_3$ — (2)

$x_1x_2 - \frac{4}{3}x_3 = 0$

using eqn (1);

$x_1^2 = \frac{4}{3}x_3$

Using eqn (2);

$x_1^2 = \frac{4}{3} \times 27$

$x_1^2 = 36$

$x_1 = \pm 6$ — (3)

From (1), (2) and (3) the possible equilibrium points are;

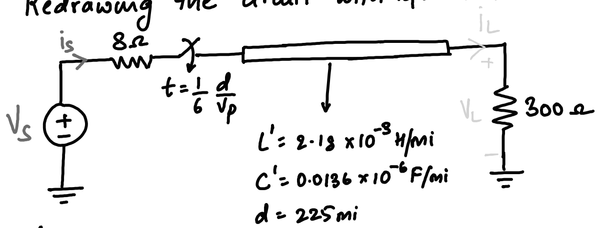
1) $x_1 = 0;\ x_2 = 0;\ x_3 = 0$

2) $x_1 = 6;\ x_2 = 6;\ x_3 = 27$

3) $x_1 = -6;\ x_2 = -6;\ x_3 = 27$

Answer 5

Redrawing the circuit with left resistance changed to 8Ω.



and

$V_s = 188{,}000 \cos(2\pi 60 t)\ V$ — (1)

We need to find V_L, i_L, i_s for the time period $0 \le t \le 0.04$ sec using a time step Δt of $\frac{1}{6}\frac{d}{v_p}$

Idea behind Bergeron model: In EMTP analysis (which involves really small time frame), the wave does not propagate instantaneously But it does so in transient stability time frame. So, the assumptions for transmission line modelling are different in both studies In the EMTP time frame since change in voltage & current at the sending end do not instantaneously affect the receiving end, the circuit can be "decoupled". And, this delay between both ends is called propagation delay. We know that electromagnetic waves travel at/close to speed of light

From L' and C' we know;

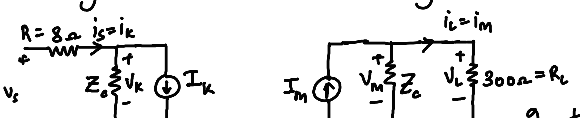
Characteristic impedance $Z_c = \sqrt{\frac{L'}{C'}} = 400.37\,\Omega$

Propagation speed $v_p = \frac{1}{\sqrt{L'C'}} = 183{,}655$ miles/s.

From the problem we know the time step $\Delta t = \frac{1}{6}\frac{d}{v_p} = 2.042 \times 10^{-4}$ s AND. propagation constant or delay $= \frac{d}{v_p} = 1225 \times 10^{-3}$ s

Using Bergeron method, the decoupled circuit is;

Sending End Model; Receiving End Model,



$I_k = i_m\left(t - \frac{d}{v_p}\right) - \frac{1}{Z_c} v_m\left(t - \frac{d}{v_p}\right)$

$I_m = i_k\left(t - \frac{d}{v_p}\right) + \frac{1}{Z_c} v_k\left(t - \frac{d}{v_p}\right)$

In terms of source & load;

$I_s(t) = i_L\left(t - \frac{d}{v_p}\right) - \frac{1}{Z_c} v_L\left(t - \frac{d}{v_p}\right)$ — (2)

$I_L(t) = i_s\left(t - \frac{d}{v_p}\right) + \frac{1}{Z_c} v_k\left(t - \frac{d}{v_p}\right)$ — (3)

Need to find: V_L, i_L and I_s. — (are all function of time).
We also know that the switch is closed at $t = \frac{1}{6}\frac{d}{v_p}$
Calculate V_L, i_L, i_s, V_s at each time step. from $0 < t < 0.04$.

→ correction in solution.

Time (t)	V_K	i_s	V_L	i_L
0	0	0	0	0
Δt $= 2.04 \times 10^{-4}$ s	$V_K = V_s - i_s \cdot R$	$i_s = \frac{V_s - V_K}{R}$	0	0
$\Delta t \times 2$ $\Delta t \times 3$ ⋮ $\Delta t \times 6$ Time step < (time at which switch closed + propagation const)	$V_K = V_s \times \frac{Z_c}{R_s + Z_c}$ $V_K(t) = \frac{Z_c}{R_s + Z_c}\left(188000 \times \cos(2\pi 60 t)\right)$	$i_s = \frac{V_s - V_K}{R}$	0 ⋮ 0	0 ⋮ 0.
$\Delta t \times 7$ Time step > (time at which switch closed + propagation const)	$V_K(t) = V_s(t) - 8 i_s(t)$ 1) solve simultaneously 2) Use $I_s(t)$ from equation ②	$i_s(t) = \frac{V_K(t)}{Z_c} + I_s(t)$	$V_L(t) = 300 \times i_L(t)$	$i_L(t) = \frac{Z_c}{300 + Z_c} \times I_L(t)$ 1) Use equation ③ for $I_L(t)$.

switch turned on at this time step at $t = \frac{1}{6}\frac{d}{v_p}$.

time at which the current and voltage waveform reaches the receiving end. Thus i_L and V_L are non-zero after this time point.

The above equations are solved for each time point and the following plots are obtained using matlab:-