Identifying Problematic AC Power Flow Alternative Solutions in Large Power Systems

Kseniia Zhgun, Farnaz Safdarian, Jack Griffin, Thomas J. Overbye
Department of Electrical and Computer Engineering
Texas A&M University
College Station, TX
{k_zhgun, fsafdarian, jackrgriffin, overbye}@tamu.edu

***Abstract*—In this paper, we propose a comprehensive algorithm for identifying alternative solutions in power flow analysis. Our approach considers various power system conditions and grid sizes, from small to large-scale cases. By analyzing the Jacobian matrix and its singularity, we accurately detect the proximity to voltage collapse and identify alternative solutions. The results demonstrate the effectiveness of the algorithm in handling diverse scenarios and showcasing its capability in identifying alternative solutions in power flow analysis.**

***Index Terms*—alternative solution, Jacobian, PV-PQ - curves, ill-conditioned system, voltage stability**

I. Introduction

The power flow (PF) analysis holds significant importance in power system applications. Achieving accurate and reliable PF results for different power system conditions is crucial for the operation and planning of power systems. However, PF equations are non-convex and nonlinear, so they can have multiple solutions [1], which makes determining the correct solution for different power system operating conditions and characteristics very challenging. These multiple solutions can be divided into normal or desired solutions, which describe the operable PF solution with voltages close to one per unit (pu), and "alternative" solutions (AS), which often exhibit abnormal voltage magnitudes far away from one pu at certain buses [2]. These AS can be categorized as low-voltage or high-voltage solutions.

Accurate identification and understanding of AS in PF analysis is highly significant. Incorrect solutions can lead to erroneous operational PF results and undesired scheduling, leading to potential operational issues and compromised system performance.

The Newton-Raphson method is widely used for solving PF equations due to its effectiveness in iterative calculations. However, this iterative method might have problems with convergence. The iterative process may diverge, indicating the absence of a solution, or it may converge to an AS instead of the desired solution. Distinguishing between these scenarios is especially challenging in heavily loaded systems [3]. Inadequate initial estimates further compound the convergence challenges. If the initial voltage solution is not accurately estimated, the iterative process may converge to an AS instead of the desired solution [4].

Extensive research has been conducted in the literature to address this problem. Several methods have been proposed to determine some or all of the PF solutions in power systems [2], [5]–[7]. Some of these approaches for determining PF solutions may not be capable of finding all possible solutions, while others that are capable may not be scalable to larger systems due to the increasing computational complexity associated with the number of power system buses.

The identification of alternative PF solutions traditionally relied on the observation of positive real parts of eigenvalues in the PF Jacobian matrix [8], [9]. But negative reactance appearance in power systems models can significantly affect the Jacobian matrix and positive real parts of eigenvalues might appear at the high-voltage operable solution [10]. In [11], authors propose a strategy for determining AS by analyzing the sensitivity of voltage magnitude to reactive power injection at each bus. However, it is important to note that the sign of dV/dQ may not always be negative in AS.

However, these methods overlook several scenarios in AS identification and there is no comprehensive study on identifying AS considering possible conditions and most references focus on small grids as case studies.

In this paper, we propose a novel and general algorithm for the identification of AS in PF analysis that considers a variety of power system conditions for large grids and is applicable for any other grids.

The structure of our paper is organized as follows. In Section II of the paper, we describe all power system characteristics considered for alternative solution identification and introduce an initial generalized algorithm. In Section III we introduce case studies. Section IV algorithm implementation results with summary and future work direction in Section V.

II. The Proposed Strategy to Determine Alternative Solutions

Typically, the PF equations can be mathematically formulated as a set of nonlinear equations, where i and k bus numbers:

$$P_i = \sum_{k=1}^{n} |V_i|\,|V_k|\,(G_{ik}cos\theta_{ik} + B_{ik}sin\theta_{ik}) \tag{1}$$

$$Q_i = \sum_{k=1}^{n} |V_i|\,|V_k|\,(G_{ik}sin\theta_{ik} - B_{ik}cos\theta_{ik}) \tag{2}$$

P_i and Q_i are the injected active and reactive power at bus, V_i bus voltage with θ_i magnitude, θ_{ik} angle between two i and k buses, G_{ik} and B_{ik} real and imaginary parts of admittance matrix Y. In PF calculations, the results include voltage magnitudes and angles at each bus, as well as the active and reactive power flow values in branches. These calculations involve considering the load demands, generation capacities, and the overall system structure as initial data [12].

A. B-Matrix

Equations 1-2 clearly demonstrate the fundamental role of the admittance matrix Y in PF calculations. The admittance matrix directly influences all PF results and is a key component in determining the voltage magnitudes, angles, and PF distribution throughout the system. The admittance matrix represents the network's components, such as transmission lines, transformers, and capacitors.

According to [11] negative reactance branches have become common in modern power system models, with up to 5% of branches exhibiting negative reactance values. The main sources of negative reactance branches include series capacitors, fictitious "star" buses in three-winding transformers, and equivalent (EQ) lines [11]. These negative reactances in the power systems influence imaginary part B of the admittance matrix Y. Even if it is present in small quantities and with low magnitudes, it can still have a significant influence on the power system characteristics. See an example Section IV, Subsection A.

In [10], it is demonstrated that systems with negative reactance can exhibit positive eigenvalues in the normal PF solution. Building upon these findings, the first step of the algorithm involves checking the B-matrix for diagonal dominance and examining the sign of diagonal elements. This step is essential in identifying potential alternative PF solutions.

B. Jacobian

Usually Newton-Raphson method is employed to solve PF, enabling the determination of voltage magnitude and angle corrections through the use of the Jacobian matrix:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial P}{\partial \theta} & \frac{\partial Q}{\partial \theta} \\ \frac{\partial P}{\partial V} & \frac{\partial Q}{\partial V} \end{bmatrix} \tag{3}$$

The Jacobian matrix plays a significant role in determining the PF solution in a system. Its characteristics, such as eigenvalues, minimum singular value, and sensitivities, provide valuable insights into the power system's condition. Indeed, the presence of negative branches in a power system can influence the characteristics of the Jacobian matrix. To accurately rely on the Jacobian matrix characteristics for solution analysis, it is essential to first understand the characteristics of the B matrix.

1) Eigenvalues: Usually, from the stability theory, we know that alternative PF solutions were typically identified based on the presence of positive eigenvalues in the PF Jacobian matrix. However, due to the potential presence of negative branches in power system cases, relying on the sign of the real part of the eigenvalues [10] is no longer a reliable method to differentiate AS cases. Therefore, in our algorithm, we exclude this step for buses where the B matrix exhibits changes corresponding to the sign of the diagonal element and broken diagonal dominance. By doing so, we account for the influence of negative branches and improve the accuracy of identifying AS in these cases.

2) Singular Values: In our algorithm, we also incorporate the minimal singular value of the PF Jacobian matrix as a key characteristic. This minimal singular value serves as a voltage security index [13], providing insights into the proximity of a specific operating point to the voltage collapse point in a power system.

By integrating the voltage security index into our identification algorithm, we can distinguish low- or high-voltage results based on their proximity to a voltage collapse point. This helps us determine if the low voltages observed are due to critical conditions in the power system or if they indicate the presence of AS results.

3) Voltage Sensitivities: Alternative PF solutions might be detected based on power system sensitivity analysis. Sensitivities represent linearized relationships that are frequently employed to quantify the impact of a small change in one variable on the rest of the power system [14]. In our algorithm, we considering sensitivities of voltages to reactive power injections in particular bus. According to [15] sensitivities for the high and low voltage solutions opposite in sign and to identify AS we can rely on negative dV/dQ sign.

$$\Delta s_{(\theta,V)} = [-J]^{-1} \cdot f_{(\theta,V)} \tag{4}$$

$$\mathbf{J}^{-1} = \begin{bmatrix} \frac{\partial \theta}{\partial P} & \frac{\partial \theta}{\partial Q} \\ \frac{\partial V}{\partial P} & \frac{\partial V}{\partial Q} \end{bmatrix} \tag{5}$$

C. Available reactive power capacity

Another important characteristic to consider in our analysis is the availability of reactive power capacity sources that may be closely associated with possible AS buses. When reactive power sources with available capacity are present, the behavior of bus sensitivities can change. As a result, the voltages, even for the AS part of the Q-V curve, will increase as the load decreases. This behavior aligns with the desired solution part of the Q-V curve. For a detailed example illustrating this behavior, please refer to Section IV, Subsection B of the paper.

D. General Algorithm

Based on the metrics discussed earlier, we propose a generalized algorithm for identifying possible alternative power flow solutions. The algorithm begins by checking the voltage magnitudes to determine if any violations occur outside the acceptable range of 0.9-1.1 pu. After checking the voltage magnitudes, the algorithm proceeds with two possible scenarios based on whether the system has negative reactance or not. This is determined by examining the imaginary part

B of the admittance matrix Y. If the system does not have negative branches, the algorithm utilizes eigenvalues and the minimum singular value to evaluate the system's proximity to a voltage collapse. This helps distinguish between low voltages caused by overloaded conditions and AS. If the characteristics mentioned above do not indicate a heavily loaded system condition, the algorithm proceeds to check the voltage sensitivities dV/dQ. As part of the algorithm, buses with negative sensitivities are identified as possible AS. If the sensitivity dV/dQ is positive, the algorithm proceeds to check the availability of reactive power sources in proximity to that bus.

Algorithm 1: The pseudo-code for identifying AS

if $Voltage\,Magnitude \leq 0.9\ and \geq 1.1$ **then**
  Check $B - matrix$:
  if $B_{ii} \leq 0$ and B has Diagonal Dominance **then**
    if $\lambda_{min} \approx 0$ **then**
      System Unstable
    else if $sign(Re(eig)) \geq$ **and** $\partial V/\partial Q \leq 0$ **then**
      Possible AS
    else if $\partial V/\partial Q \geq 0$ **then**
      Check Available Q-capacity:
      if Close to Available Q-capacity **then**
        Possible AS
      end if
    end if
  else if $B_{ii} \geq 0$ and lost Diagonal Dominance **then**
    if $\lambda_{min} \approx 0$ **then**
      System Unstable
    else
      Possible AS
    end if
  end if
end if=0

III. Case Studies

To justify the algorithm steps, we consider small-scale grids such as the 5-bus and 19-bus grids. These grids allow for thorough validation and verification of the algorithm's effectiveness in detecting alternative PF solutions. Additionally, we provide an example of implementing the algorithm on a larger-scale system with a 24k-bus grid. By applying the algorithm to a large-scale realistic power system, we demonstrate its scalability to more complex systems. To be able to calculate Jacobian's eigenvalues, singular values, and sensitivity characteristics we consider it as a sparse matrix for more efficient calculations. The combination of small-scale grids and large-scale grids ensures that the algorithm is rigorously tested and can be applied to a range of power system scenarios.

A. Five-Bus Grid

Figure 1 shows more details of this five-bus grid. To assess the impact of negative reactance on the five-bus system, we analyze the presence of negative branches in several cases shown in Table I.

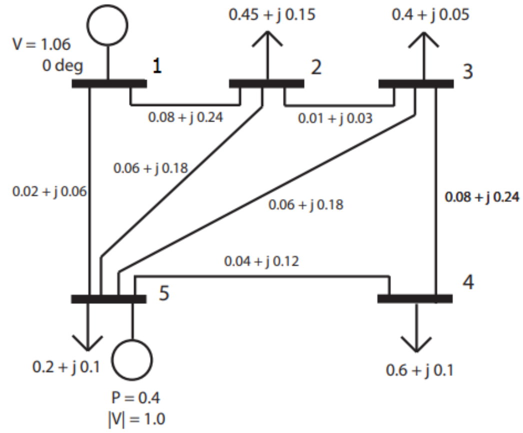

Fig. 1. Five-bus grid [16]

TABLE I
Reactance for 5 bus Grid

Branch	Case 1	Case 2	Case 3	Case 4
Reactance, Ohms				
1-2	0.24	0.24	0.24	0.24
2-3	0.03	0.03	-0.03	-0.03
3-4	0.24	-0.24	-0.24	-0.24
1-5	0.06	0.06	0.06	-0.06
5-4	0.12	0.12	0.12	0.12
5-3	0.18	0.18	0.18	0.18
5-2	0.18	0.18	0.18	0.18

B. EPRI 19-Bus Grid

Figure 2 and Table II show important characteristics of this 19-bus EPRI grid [17]. Two scenarios including Case 1 without negative branches and Case 2 with one negative branch are studied. For these two cases, we analyze the Q-V curve [18] obtained by incrementally increasing the load.

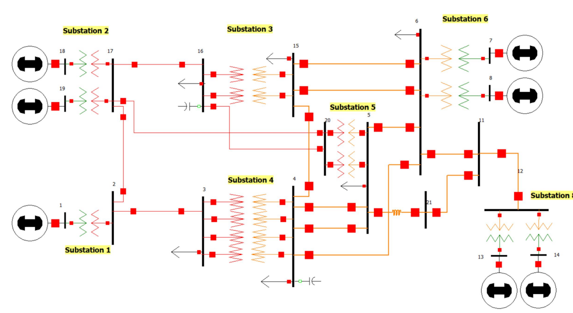

Fig. 2. EPRI 19-bus grid

TABLE II
19-bus Grid Statistics

Parameter	Numerical Value
Number of buses	19
Number of generators	7
Number of loads	6
Number of switched shunts	2
Number of substations	8
Number of transmission lines	15

C. Midwest 24k-Bus Synthetic Grid

The larger case study is Midwest 24k-bus synthetic grid over the US Midwest [19]. The transmission network is built based on the actual transmission voltage levels in this area, including 500 kV, 345 kV, 230 kV, 161 kV, 138 kV, 115 kV, and 69 kV. Fig. 3 shows the one-line diagram on the transmission grid for Midwest 24k-bus synthetic grid where 500 kV lines and 345 kV are in green, 230 blue, 161, 138, 115, and 69 kV black. Table III provides a summary of the case.

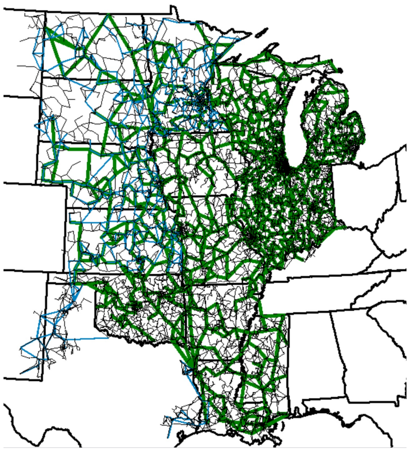

Fig. 3. Transmission lines in the Midwest 24k-bus synthetic grid

TABLE III
MIDWEST 24K-BUS CASE STATISTICS

Parameter	Numerical Value
Number of buses	23,643
Number of generators	6,274
Number of loads	11,731
Number of switched shunts	1,218
Number of substations	14,069
Number of transmission lines	23,787
Total design load (MW)	202,000
Total design generation (MW)	321,680

IV. RESULTS

In this section, we present the results of the study by describing each step of the proposed algorithm in detail. By providing comprehensive explanations and a variety of grid sizes, we aim to showcase the applicability and robustness of our approach in identifying alternative PF solutions.

We begin by outlining the specific steps of the algorithm, highlighting the rationale behind each step and its role in identifying AS. We provide a clear description of the algorithm's workflow, ensuring a thorough understanding of its functioning.

A. Five-Bus Grid

After checking the voltage violation in the algorithm's line 1 and in case it was out of range, so we look at B matrix diagonal elements sign and diagonal dominance (lines 2-3 and 11).

In Case 1 all branches have a positive reactance, matrix B diagonal elements are negative with the diagonal dominance. The real parts of its eigenvalues are negative which means that system in a normal condition.

$$B = \begin{bmatrix} -18.75 & 3.75 & 0 & 0 & 15.00 \\ 3.75 & -38.75 & 30.00 & 0 & 5.00 \\ 0 & 30.00 & -38.75 & 3.75 & 5.00 \\ 0 & 0 & 3.75 & -11.25 & 7.50 \\ 15.00 & 5.00 & 5.00 & 7.50 & -32.50 \end{bmatrix}$$

For the same case, we consider different solutions evaluated with homotopy method [20]. The first solution is the high-voltage solution, with voltage results close to nominal and small angles, all eigenvalues have negative real parts. Second solution belongs to the Type-1 solution, which has one positive real part of the eigenvalue. And the third solution with two real-part eigenvalues (Table IV).

We can see that for case without negative branches we can identify alternative solutions according to the real-part eigenvalue sign.

TABLE IV
VOLTAGE RESULTS AND EIGENVALUES FOR FIVE-BUS SYSTEM CASE 1

Bus	Solution 1	Solution 2	Solution 3
		Voltage	
1	1.060∠0.00	1.060∠0.00	1.060∠0.00
2	0.980∠-4.53	0.034∠-69.0	0.196∠-30.70
3	0.977∠-4.85	0.184∠-37.81	0.036∠-85.96
4	0.966∠-5.69	0.686∠-23.87	0.081∠-79.42
5	1.000∠-2.06	1.000∠-16.91	1.000 ∠-22.52
n		Eigenvalues	
1	-66.69 + 13.19i	-21.55 + 0.00i	-14.42 + 0.00i
2	-66.69 + 13.19i	5.55 + 0.00i	3.69 + 0.00i
3	-36.37 + 0.00i	-8.10 + 0.00i	2.23 + 0.00i
4	-0.97 + 0.00i	-6.26 + 0.00i	-4.12 + 3.36i
5	-17.24 + 0.00i	-2.41 + 0.86i	-4.12 - 3.36i
6	-7.60 + 0.00i	-2.41 - 0.86i	-4.12 + 0.00i
7	-8.89 + 0.00i	-3.26 + 0.00i	-1.00 + 0.00i

In case 2, one negative branch changes the diagonal dominance for row 4, all eigenvalues (Table V) have negative real part.

$$B = \begin{bmatrix} -18.75 & 3.75 & 0 & 0 & 15.00 \\ 3.75 & -38.75 & 30.00 & 0 & 5.00 \\ 0 & 30.00 & -31.25 & -3.75 & 5.00 \\ 0 & 0 & -3.75 & -3.75 & 7.50 \\ 15.00 & 5.00 & 5.00 & 7.50 & -32.50 \end{bmatrix}$$

Case 3 has two negative branches and we can see not only changes in diagonal dominance, bus changes in the sign of diagonal elements for the second and third rows, real part of two eigenvalues became positive.

$$B = \begin{bmatrix} -18.75 & 3.75 & 0 & 0 & 15.00 \\ 3.75 & 21.25 & -30.00 & 0 & 5.00 \\ 0 & -30.00 & 28.75 & -3.75 & 5.00 \\ 0 & 0 & -3.75 & -3.75 & 7.50 \\ 15.00 & 5.00 & 5.00 & 7.50 & -32.50 \end{bmatrix}$$

Case 4 has three negative branches and all diagonal elements lost their dominance and three of them change the sign. In this case, we got three eigenvalues with positive real parts.

$$B = \begin{bmatrix} 11.25 & 3.75 & 0 & 0 & -15.00 \\ 3.75 & 21.25 & -30.00 & 0 & 5.00 \\ 0 & -30.00 & 28.75 & -3.75 & 5.00 \\ 0 & 0 & -3.75 & -3.75 & 7.50 \\ -15.00 & 5.00 & 5.00 & 7.50 & -2.50 \end{bmatrix}$$

TABLE V
VOLTAGES RESULT AND EIGENVALUES FOR FIVE-BUS SYSTEM CASE 2-4

Bus	Case 2	Case 3	Case 4
	Voltage		
1	1.06∠0.00	1.06∠0.00	1.06∠0.00
2	0.99∠-4.80	1.00∠-4.93	0.98∠-1.23
3	0.99∠-5.20	0.99∠-4.67	0.98∠-1.23
4	0.93∠-5.20	0.92∠ -5.45	0.95∠ -0.23
5	1.00∠-2.08	1.00∠-2.08	1.00∠2.88
n	Eigenvalues		
1	-64.8 + 22.6i	55.1 + 22.8i	54.2 + 22.4i
2	-64.8 - 22.6i	55.1 - 22.8i	54.2 - 22.4i
3	-36.1 + 0.0i	-36.1 + 0.0i	6.0 + 0.0i
4	-1.3 + 4.1i	-1.8 + 3.8i	-1.9 + 4.0i
5	-1.3 - 4.1i	-1.8 + 3.8i	-1.9 - 4.0i
6	-3.8 + 0.0i	-3.9 + 0.0i	-11.8 + 0.0i
7	-6.9 + 0.0i	-6.8 + 0.0i	-9.4 + 0.0i

We can summarize these results with the conclusion that one of the first steps of identifying algorithm should be a B matrix checking to see if there are enough changes in the matrix structure to change Jacobian and eigenvalues.

B. EPRI 19-Bus Grid

Typically, in power grids, when the system is approaching voltage collapse, it is easier for PF to converge to an AS due to the proximity of neighboring high-voltage solution and AS.

In lines 4 and 12 in the algorithm, we are looking at the minimal singular value to understand how far we are from the voltage collapse point. This Jacobian characteristic can help us to understand if low-voltages in PF solutions appear because of stressed system conditions or if this is an AS.

In case 1, there are no negative branches and we consider different scenarios including a normal solution far from collapse (point 1), an AS far from collapse (point 2), a normal solution close to collapse (point 3), and an AS close to collapse (point 4). These points are shown in Figure 4.

As shown in Table VI, we are using several parameters such as minimal singular value to recognize if the system is heavily loaded and we can differentiate between different types of solutions. After that, in line 5 of our algorithm, we look at the eigenvalues and dV/dQ sign if the B matrix diagonal dominance is not broken and the diagonal elements are negative.

For this case, we can summarize that combining low voltage, high minimal singular value, and sensitivity analysis can serve as reliable indicators for identifying AS.

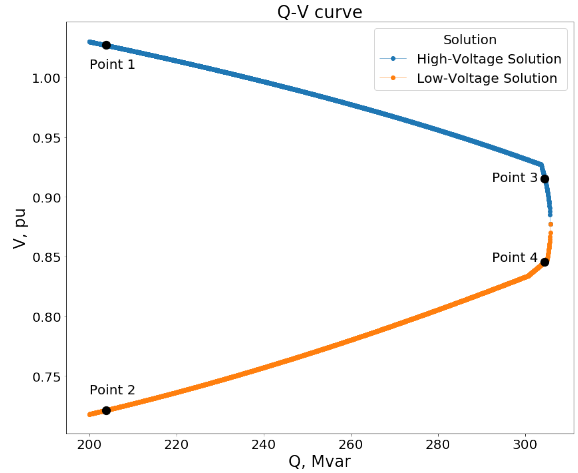

Fig. 4. 19-bus system without negative branches - bus 3

TABLE VI
SINGULAR VALUES, EIGENVALUES AND SENSITIVITY

Par./Sc.	1	2	3	4
λ_{min}	0.449	1.000	0.0174	0.0177
$sign(Re(Eig))$	all negative	1 positive	all negative	1 positive
diag(dV/dQ)	all positive	7 negative	all positive	25 negative

Case 2 of this 19-bus grid has one negative branch and highlights that relying solely on the sign of dV/dQ (Table VII) is not always a dependable criterion for identifying whether a system is on the AS curve. Point 1 on the AS curve in Figure 5 with $Q = 278.3\,Mvar$.

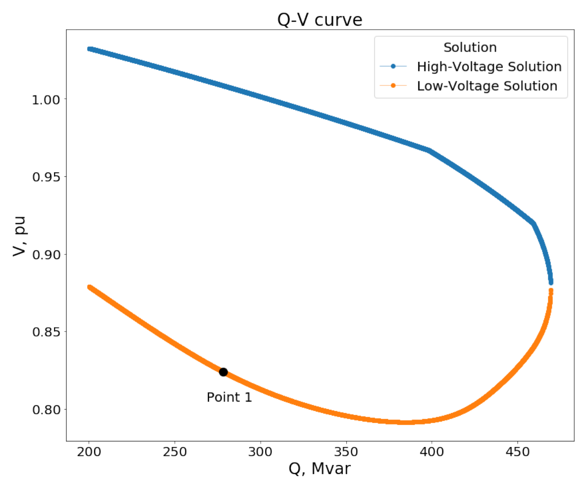

Fig. 5. 19-bus system with one negative branch - bus 3

TABLE VII
SINGULAR VALUES, EIGENVALUES AND BUS SENSITIVITY

State	Singular Value	sign(Re(Eig))	dV/dQ
λ_{min}	0.035	5 pos.	0.0117

C. Midwest 24k-Bus Synthetic Grid

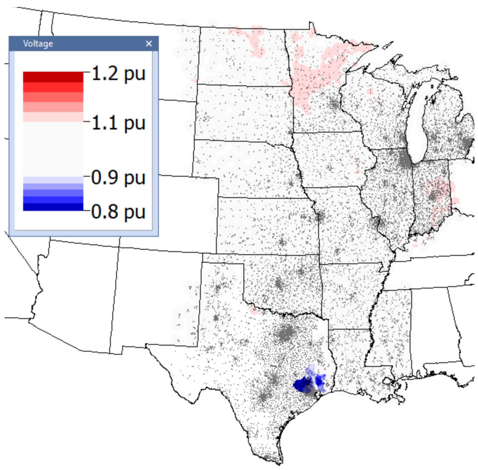

Fig. 6. Voltage contour of Midwest 24k-bus synthetic grid

For the large-scale case we consider Midwest 24k-bus synthetic grid. This grid consists of several negative reactances.

With our algorithm, we are starting with the first step - line 1, 80 buses have a voltage level lower than 0.9 and some of them have significantly low voltage. Figure 6 shows a voltage contour of this case based on the strategy explained in [21].

For example, the voltage of Bus 12165 is 0.09579 pu.

The next step is to check the B matrix as line 2 of the algorithm.

$B_{12165,12165} = -22.3581$ Diagonal element is negative, but diagonal dominance is broken:

$|B_{12165,12165}| - \sum_{12165 \neq j}^{n} |B_{12165,j}| = -0.0070 \leq 0$ that is why we are going to the line 12 of the algorithm and checking the minimal singular value which is $\lambda_{min} = 0.0002$. Therefore, the system is close to voltage collapse but still in a stable condition. So for this bus, our algorithm identifies it as a possible AS.

V. Conclusions and Future Work

This paper provided a generalized algorithm to identify possible AS in PF analysis for various cases. This algorithm considers different power system conditions such as changes in the imaginary part of the admittance matrix, due to negative branches, and after that looks at Jacobean's eigenvalues, minimal singular values, and sensitivities. For this algorithm, we justified every step with examples and showed implementations on a large grid.

Future work will focus on expanding the algorithm by incorporating additional characteristics related to both voltage magnitudes and voltage angles. Additionally, efforts will be made to investigate methods for transitioning from AS to the desired normal PF solution.

VI. Acknowledgements

This work was partially supported through funding provided by Texas A&M Engineering Experiment Station (TEES) Smart Grid Center.

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