ECEN 667
Power System Stability

Lecture 21: Oscillations, Measurement-Based Modal Analysis

Prof. Tom Overbye
Dept. of Electrical and Computer Engineering
Texas A&M University
overbye@tamu.edu

Announcements

- Read Chapter 8
- Homework 6 is due on Tuesday Nov 21

Small Signal Stability Analysis

- Small signal stability is the ability of the power system to maintain synchronism following a small disturbance
  - System is continually subject to small disturbances, such as changes in the load
- The operating equilibrium point (EP) obviously must be stable
- Small system stability analysis (SSA) is studied to get a feel for how close the system is to losing stability and to get additional insight into the system response
  - There must be positive damping

Model Based SSA

- Assume the power system is modeled in our standard form as

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{y}\right)$$

$$\mathbf{0} = \mathbf{g}(\mathbf{x}, \mathbf{y})$$

- The system can be linearized about an equilibrium point

$$\Delta\dot{\mathbf{x}} = \mathbf{A}\Delta\mathbf{x} + \mathbf{B}\Delta\mathbf{y}$$

$$\mathbf{0} = \mathbf{C}\Delta\mathbf{x} + \mathbf{D}\Delta\mathbf{y}$$

If there are just classical generator models then **D** is the power flow Jacobian; otherwise it also includes the stator algebraic equations

- Eliminating $\Delta\mathbf{y}$ gives

$$\Delta\dot{\mathbf{x}} = \left(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right)\Delta\mathbf{x} = \mathbf{A}_{sys}\Delta\mathbf{x}$$

Modal Analysis - Comments

- The matrix $\mathbf{A}_{sys}$ can be calculated doing a partial factorization, just like what is done with Kron reduction in creating power system equivalents
- SSA is done by looking at the eigenvalues (and other properties) of $\mathbf{A}_{sys}$
- Modal analysis (analysis of small signal stability through eigenvalue analysis) is at the core of SSA software
- In Modal Analysis one looks at:
  - Eigenvalues, Eigenvectors (left or right)
  - Participation factors
  - Mode shape
- Power System Stabilizer (PSS) design in a multi-machine context can be done using the modal analysis method.

Goal is to determine how the various parameters affect the response of the system

Eigenvalues, Right Eigenvectors

- For an n by n matrix **A** the eigenvalues of **A** are the roots of the characteristic equation:

$$\det[\mathbf{A} - \lambda\mathbf{I}] = \left|\mathbf{A} - \lambda\mathbf{I}\right| = 0$$

- Assume $\lambda_1 \ldots \lambda_n$ as distinct (no repeated eigenvalues).
- For each eigenvalue λ_i there exists an eigenvector such that:

$$\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

- $\mathbf{v}_i$ is called a right eigenvector
- If λ_i is complex, then $\mathbf{v}_i$ has complex entries

Left Eigenvectors, Eigenvector Properties

- For each eigenvalue λ_i there exists a left eigenvector $\mathbf{w}_i$ such that:
  $$\mathbf{w}_i^t \mathbf{A} = \mathbf{w}_i^t \lambda_i$$
- Equivalently, the left eigenvector is the right eigenvector of $\mathbf{A}^T$; that is,
  $$\mathbf{A}^t \mathbf{w}_i = \lambda_i \mathbf{w}_i$$
- The right and left eigenvectors are orthogonal i.e.
  $$\mathbf{w}_i^t \mathbf{v}_i \neq 0 \ , \ \mathbf{w}_i^t \mathbf{v}_j = 0 \quad (i \neq j)$$
- We can normalize the eigenvectors so that:
  $$\mathbf{w}_i^t \mathbf{v}_i = 1 \ , \ \mathbf{w}_i^t \mathbf{v}_j = 0 \quad (i \neq j)$$

Eigenvector Example

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}, \ |\mathbf{A} - \lambda \mathbf{I}| = 0 \ \Rightarrow \ \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 3\lambda - 10 = 0 \Rightarrow \ \lambda_{1,2} = \frac{3 \pm \sqrt{(3)^2 + 4(10)}}{2} = \frac{3 \pm \sqrt{49}}{2} = 5, -2$$

Right Eigenvectors $\lambda_1 = 5$

$$\mathbf{A}\mathbf{v}_1 = 5\mathbf{v}_1 \Rightarrow \ \mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} \Rightarrow \begin{matrix} v_{11} + 4v_{21} = 5v_{11} \\ 3v_{11} + 2v_{21} = 5v_{21} \end{matrix}$$

$$\textit{choose } \mathbf{v}_{21} = 1 \Rightarrow \mathbf{v}_{11} = 1$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Similarly,

$$\lambda_2 = -2 \Rightarrow \ \mathbf{v}_2 = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

Eigenvector Example

- Left eigenvectors

$$\lambda_1 = 5 \ \mathbf{w}_1^t \mathbf{A} = \mathbf{w}_1^t 5 \Rightarrow [w_{11} \ \ w_{21}] \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} = 5[w_{11} \ \ w_{21}]$$

$$\begin{matrix} w_{11} + 3w_{21} = 5w_{11} \\ 4w_{11} + 2w_{21} = 5w_{21} \end{matrix} \Rightarrow \mathit{Let} \ w_{21} = 4, \ \mathit{then} \ \ w_{11} = 3$$

$$\mathbf{w}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \lambda_2 = -2 \Rightarrow \mathbf{w}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ \ \mathbf{v}_2 = \begin{bmatrix} 4 \\ -3 \end{bmatrix} \ \ \mathbf{w}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \ \ \mathbf{w}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathit{Verify} \ \ \mathbf{w}_1^t \mathbf{v}_1 = 7 \ , \ \mathbf{w}_2^t \mathbf{v}_2 = 7 \ , \ \mathbf{w}_2^t \mathbf{v}_1 = 0 \ , \ \mathbf{w}_1^t \mathbf{v}_2 = 0$$

We would like to make $w_i^t v_i = 1$.

This can be done in many ways.

Eigenvector Example

$$\text{Let} \quad \mathbf{W} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}$$

$$\text{Then} \quad \mathbf{W}^T\mathbf{V} = \mathbf{I}$$

$$\text{Verify} \quad \frac{1}{7}\begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- It can be verified that $\mathbf{W}^T = \mathbf{V}^{-1}$.
- The left and right eigenvectors are used in computing the participation factor matrix.

Modal Matrices

- The deviation away from an equilibrium point can be defined as

  $\Delta\dot{\mathbf{x}} = \mathbf{A}\Delta\mathbf{x}$

- From this equation it is difficult to determine how parameters in **A** affect a particular **x** because of the variable coupling
- To decouple the problem first define the matrices of the right and left eigenvectors (the modal matrices)

$$\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2 \ldots\ldots \mathbf{v}_n] \ \& \ \mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, \ldots\ldots \mathbf{w}_n]$$

$$\mathbf{AV} = \mathbf{V\Lambda} \quad \text{when} \quad \mathbf{\Lambda} = Diag(\lambda_i)$$

V represents the right eigenvectors

Modal Matrices

- It follows that

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{\Lambda}$$

- To decouple the variables define $\mathbf{z}$ so

$$\Delta\mathbf{x} = \mathbf{V}\mathbf{z} \rightarrow \Delta\dot{\mathbf{x}} = \mathbf{V}\dot{\mathbf{z}} = \mathbf{A}\Delta\mathbf{x} = \mathbf{A}\mathbf{V}\mathbf{z}$$

- Then

$$\dot{\mathbf{z}} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}\mathbf{z} = \mathbf{W}\mathbf{A}\mathbf{V}\mathbf{z} = \mathbf{\Lambda}\mathbf{z}$$

- Since Λ is diagonal, the equations are now uncoupled with

$$\dot{z}_i = \lambda_i z_i$$

- So $\Delta\mathbf{x}(t) = \mathbf{V}\mathbf{z}(t)$

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} 1 & 4 \\ 1 & -3 \end{bmatrix}$$

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$$

Modal Matrices

- Thus the response can be written in terms of the individual eigenvalues and right eigenvectors as

$$\Delta \mathbf{x}(t) = \sum_{i=1}^{n} \mathbf{v}_i z_i(0) e^{\lambda_i t}$$

Note, we are requiring that the eigenvalues be distinct!

- Furthermore with

$$\Delta \mathbf{x} = \mathbf{V}\mathbf{Z} \Rightarrow \quad \mathbf{z} = \mathbf{V}^{-1}\mathbf{x} = \mathbf{W}^{T}\mathbf{x}$$

- So z(t) can be written as using the left eigenvectors as

$$\mathbf{z}(t) = \mathbf{W}^{t}\mathbf{x}(t) = [\mathbf{w}_1 \;\; \mathbf{w}_2 \ldots . \mathbf{w}_n]^{t} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Modal Matrices

- We can then write the response x(t) in terms of the modes of the system

$$z_i(t) = w_i^t x(t)$$

$$z_i(0) = w_i^t x(0) \triangleq c_i$$

$$\text{so} \quad \mathbf{x}(t) = \sum_{i=1}^{n} \mathbf{v}_i c_i e^{\lambda_i t}$$

$$\text{Expanding} \quad \Delta x_i(t) = v_{i1} c_1 e^{\lambda_1 t} + v_{i2} c_2 e^{\lambda_2 t} + \ldots v_{in} c_n e^{\lambda_n t}$$

- So c_i is a scalar that represents the magnitude of excitation of the i^{th} mode from the initial conditions

Numerical example

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}, \ \Delta\mathbf{x}(0) = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

Eigenvalues are $\lambda_1 = -4$, $\lambda_2 = 2$

Eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Modal matrix $\mathbf{V} = \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix}$

Normalize so $\mathbf{V} = \begin{bmatrix} 0.2425 & 0.4472 \\ -0.9701 & 0.8944 \end{bmatrix}$

Left eigenvector matrix is:

$$\mathbf{W}^{\mathbf{T}} = \mathbf{V}^{-1} = \begin{bmatrix} 1.3745 & -0.6872 \\ 1.4908 & 0.3727 \end{bmatrix}$$

$$\dot{\mathbf{z}} = \mathbf{W}^{\mathbf{T}}\mathbf{A}\mathbf{V}\mathbf{z}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Numerical example (contd)

$$\dot{z}_1 = -4z_1 \ , \ \mathbf{z}(0) = V^{-1}\mathbf{x}(0)$$

$$\dot{z}_2 = 2z_2 \ , \ \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} 4.123 \\ 0 \end{bmatrix}$$

$$z_1(t) = z_1(0)e^{-4t} \ ; \ z_2(t) = z_2(0)e^{2t}, \mathbf{C} = \mathbf{W}^T\mathbf{x}(0) = \begin{bmatrix} 4.123 \\ 0 \end{bmatrix}$$

$$\mathbf{x} = \mathbf{V}\mathbf{z}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

$$= c_1 \begin{bmatrix} 0.2425 \\ -0.9701 \end{bmatrix} z_1(t) + c_2 \begin{bmatrix} 0.4472 \\ 0.8944 \end{bmatrix} z_2(t) = \sum_{i=1}^{2} c_i \mathbf{v}_i z_i(0) e^{\lambda_i t}$$

Because of the initial condition, the 2nd mode does not get excited

Mode Shape, Sensitivity and Participation Factors

- So we have

$$\mathbf{x}(t) = \mathbf{V}\mathbf{z}(t), \quad \mathbf{z}(t) = \mathbf{W}^t\mathbf{x}(t)$$

- **x**(t) are the original state variables, **z**(t) are the transformed variables so that each variable is associated with only one mode.
- From the first equation the right eigenvector gives the "mode shape" i.e. relative activity of state variables when a particular mode is excited.
- For example the degree of activity of the state variable x_k in the $\mathbf{v}_i$ mode is given by the element V_{ki} of the right eigenvector matrix **V**

Mode Shape, Sensitivity and Participation Factors

- The magnitude of elements of $\mathbf{v}_i$ give the extent of activities of *n* state variables in the i^{th} mode and angles of elements (if complex) give phase displacements of the state variables with regard to the mode.
- The left eigenvector $\mathbf{w}_i$ identifies which combination of original state variables display only the i^{th} mode.

Eigenvalue Parameter Sensitivity

- To derive the sensitivity of the eigenvalues to the parameters recall $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$; take the partial derivative with respect to A_{kj} by using the chain rule

$$\frac{\partial \mathbf{A}}{\partial \mathbf{A}_{kj}} \mathbf{v}_i + \mathbf{A} \frac{\partial \mathbf{v_i}}{\partial A_{kj}} = \frac{\partial \lambda_i}{\partial A_{kj}} \mathbf{v}_i + \lambda_i \frac{\partial \mathbf{v}_i}{\partial A_{kj}}$$

Multiply by $\mathbf{w}_i^t$

$$\mathbf{w}_i^t \frac{\partial \mathbf{A}}{\partial \mathbf{A}_{kj}} \mathbf{v}_i + \mathbf{w}_i^t \mathbf{A} \frac{\partial \mathbf{v_i}}{\partial A_{kj}} = \mathbf{w}_i^t \frac{\partial \lambda_i}{\partial A_{kj}} \mathbf{v}_i + \mathbf{w}_i^t \lambda_i \frac{\partial \mathbf{v}_i}{\partial A_{kj}}$$

$$\mathbf{w}_i^t \frac{\partial \mathbf{A}}{\partial A_{kj}} \mathbf{v}_i + \mathbf{w}_i^t [\mathbf{A} - \lambda_i \mathbf{I}] \frac{\partial \mathbf{v}_i}{\partial A_{kj}} = \mathbf{w}_i^t \frac{\partial \lambda_i}{\partial A_{kj}} \mathbf{v}_i$$

Eigenvalue Parameter Sensitivity

- This is simplified by noting that $\mathbf{w}_i^t(\mathbf{A} - \lambda_i \mathbf{I}) = 0$ by the definition of $\mathbf{w}_i$ being a left eigenvector
- Therefore

$$\mathbf{w}_i^t \frac{\partial \mathbf{A}}{\partial A_{kj}} \mathbf{v}_i = \frac{\partial \lambda_i}{\partial A_{kj}}$$

- Since all elements of $\frac{\partial \mathbf{A}}{\partial A_{kj}}$ are zero, except the k^{th} row, j^{th} column is 1
- Thus

$$\frac{\partial \lambda_i}{\partial A_{kj}} = W_{ki} V_{ji}$$

Sensitivity Example

- In the previous example we had

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}, \quad \lambda_{1,2} = 5, -2, \quad \mathbf{V} = \begin{bmatrix} 1 & 4 \\ 1 & -3 \end{bmatrix}, \quad \mathbf{W} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}$$

- Then the sensitivity of λ_1 and λ_2 to changes in **A** are

$$\frac{\partial \lambda_i}{\partial A_{kj}} = W_{ki} V_{ji} \rightarrow \frac{\partial \lambda_1}{\partial \mathbf{A}} = \frac{1}{7}\begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}, \quad \frac{\partial \lambda_2}{\partial \mathbf{A}} = \frac{1}{7}\begin{bmatrix} 4 & -3 \\ -4 & 3 \end{bmatrix}$$

- For example with $\hat{\mathbf{A}} = \begin{bmatrix} 1 & 4 \\ 3 & 2.1 \end{bmatrix}, \quad \hat{\lambda}_{1,2} = 5.057, -1.957$
- Or if

$$\hat{\mathbf{A}} = \begin{bmatrix} 1 & 4 \\ 3 & 3 \end{bmatrix}, \quad \hat{\lambda}_{1,2} = 5.61, -1.61,$$

Participation Factors

- The participation factors, P_{ki}, are used to determine how much the k^{th} state variable participates in the i^{th} mode

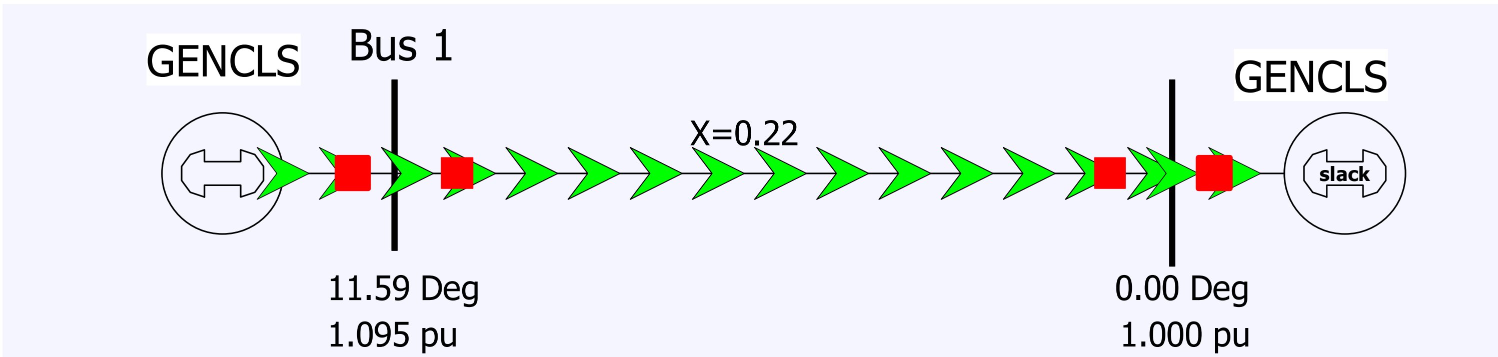
$$P_{ki} = V_{ki} W_{ki}$$

- The sum of the participation factors for any mode or any variable sum to 1
- The participation factors are quite useful in relating the eigenvalues to portions of a model
- For the previous example with $P_{ki} = V_{ki} W_{ik}$ and

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} 1 & 4 \\ 1 & -3 \end{bmatrix}, \quad \mathbf{W} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix} \longrightarrow \mathbf{P} = \frac{1}{7}\begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$$

SSA Two Generator Example

- Consider the two bus, two classical generator system from lectures 18 and 20 with $X_{d1}'=0.3$, $H_1=3.0$, $X_{d2}'=0.2$, $H_2=6.0$



- Essentially everything needed to calculate the **A**, **B**, **C** and **D** matrices was covered in lecture 15

SSA Two Generator Example

- The **A** matrix is calculated differentiating **f**(**x**,**y**) with respect to **x** (where **x** is δ_1, $\Delta\omega_1$, δ_2, $\Delta\omega_2$)

$$\frac{d\delta_1}{dt} = \Delta\omega_{1.pu}\omega_s$$

$$\frac{d\Delta\omega_{1,pu}}{dt} = \frac{1}{2H_1}\left(P_{M1} - P_{E1} - D_1\Delta\omega_{1.pu}\right)$$

$$\frac{d\delta_2}{dt} = \Delta\omega_{2.pu}\omega_s$$

$$\frac{d\Delta\omega_{2,pu}}{dt} = \frac{1}{2H_2}\left(P_{M2} - P_{E2} - D_2\Delta\omega_{1.pu}\right)$$

$$\mathrm{P}_{Ei} = \left(E_{Di}^2 - E_{Di}V_{Di}\right)G_i + \left(E_{Qi}^2 - E_{Qi}V_{Qi}\right)G_i + \left(E_{Di}V_{Qi} - E_{Qi}V_{Di}\right)B_i$$

$$E_{Di} + jE_{Qi} = E_i'\left(\cos\delta_i + j\sin\delta_i\right)$$

SSA Two Generator Example

- Giving

$$\mathbf{A} = \begin{bmatrix} 0 & 376.99 & 0 & 0 \\ -0.761 & 0 & 0 & 0 \\ 0 & 0 & 0 & 376.99 \\ 0 & 0 & -0.389 & 0 \end{bmatrix}$$

- **B**, **C** and **D** are as calculated previously for the implicit integration, except the elements in B are not multiplied by $\Delta t/2$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.2889 & 0.6505 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0833 & 0.3893 \end{bmatrix}$$

SSA Two Generator Example

- The **C** and **D** matrices are

$$\mathbf{C} = \begin{bmatrix} -3.903 & 0 & 0 & 0 \\ -1.733 & 0 & 0 & 0 \\ 0 & 0 & -4.671 & 0 \\ 0 & 0 & 1.0 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 7.88 & 0 & -4.54 \\ -7.88 & 0 & 4.54 & 0 \\ 0 & -4.54 & 0 & 9.54 \\ 4.54 & 0 & -9.54 & 0 \end{bmatrix}$$

- Giving

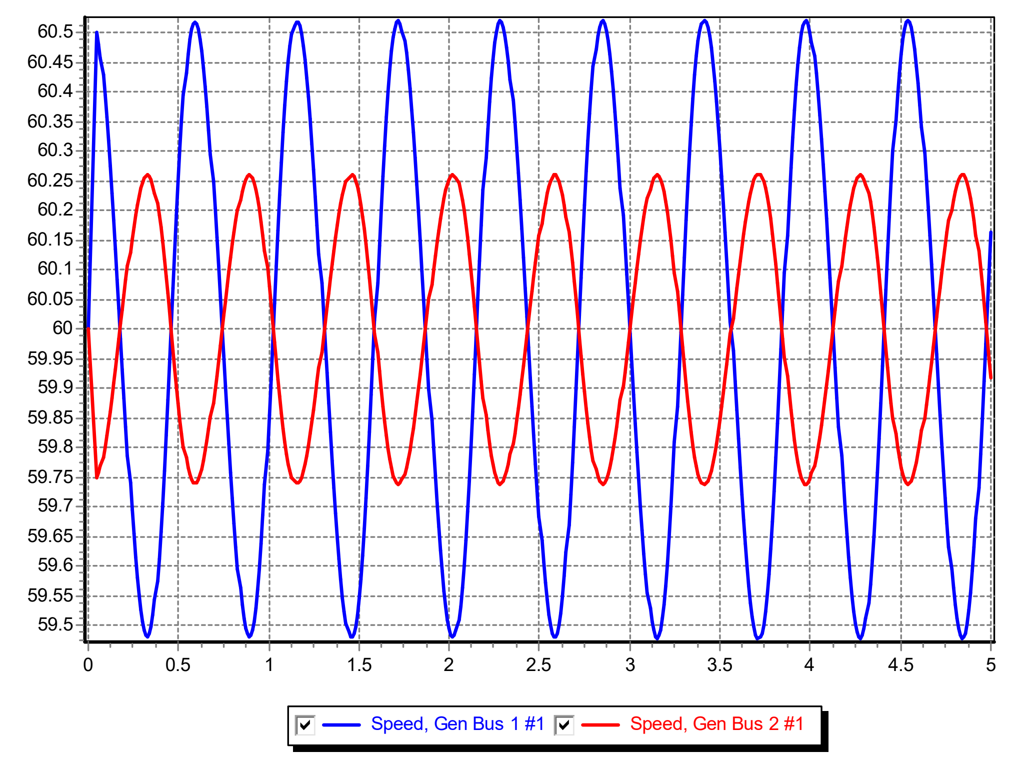
$$\mathbf{A}_{sys} = \mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C} = \begin{bmatrix} 0 & 376.99 & 0 & 0 \\ -0.229 & 0 & 0.229 & 0 \\ 0 & 0 & 0 & 376.99 \\ 0.114 & 0 & -0.114 & 0 \end{bmatrix}$$

SSA Two Generator

- Calculating the eigenvalues gives a complex pair and two zero eigenvalues
- The complex pair, with values of +/- j11.39 corresponds to the generators oscillating against each other at 1.81 Hz
- One of the zero eigenvalues corresponds to the lack of an angle reference
  - Could be rectified by redefining angles to be with respect to a reference angle (see book 226) or we just live with the zero
- Other zero is associated with lack of speed dependence in the generator torques

SSA Two Generator Speeds

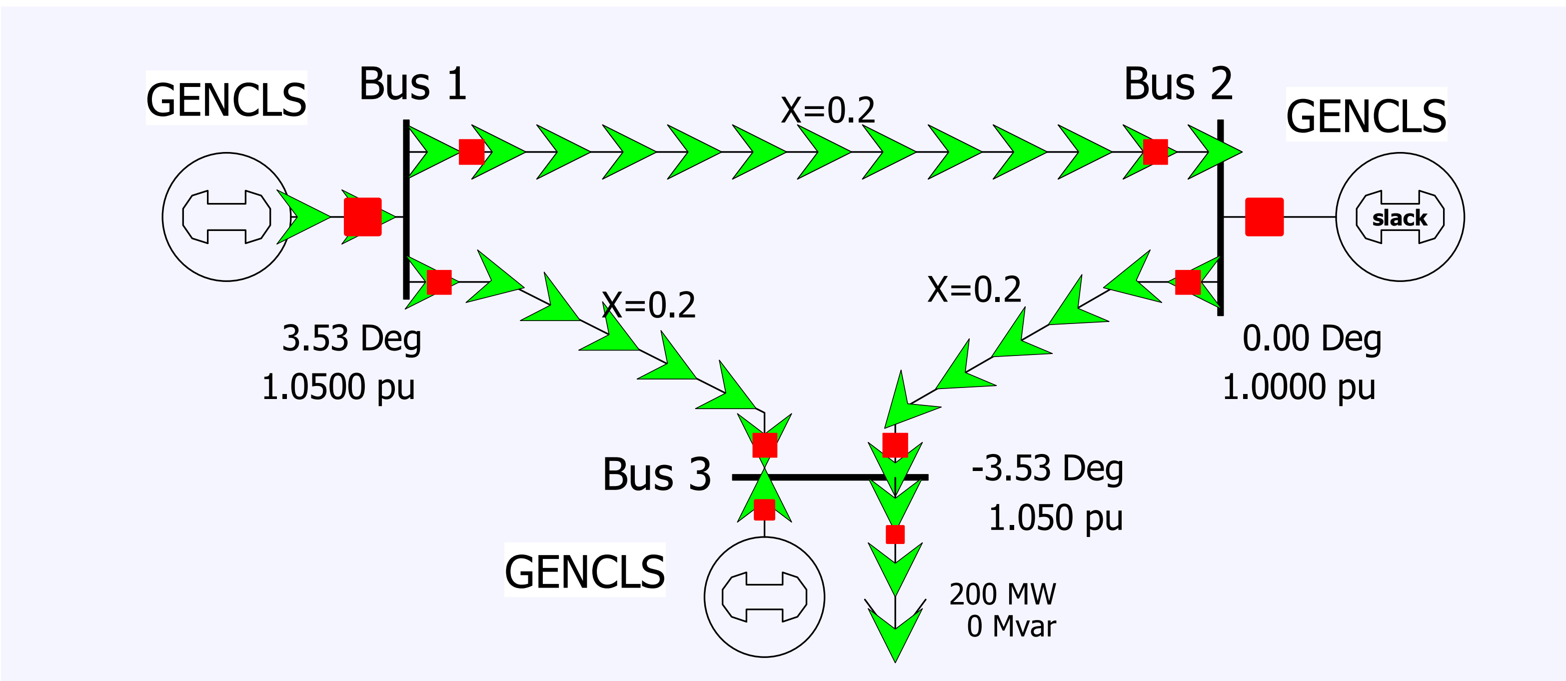
- The two generator system response is shown below for a small disturbance



Notice the actual response closely matches the calculated frequency

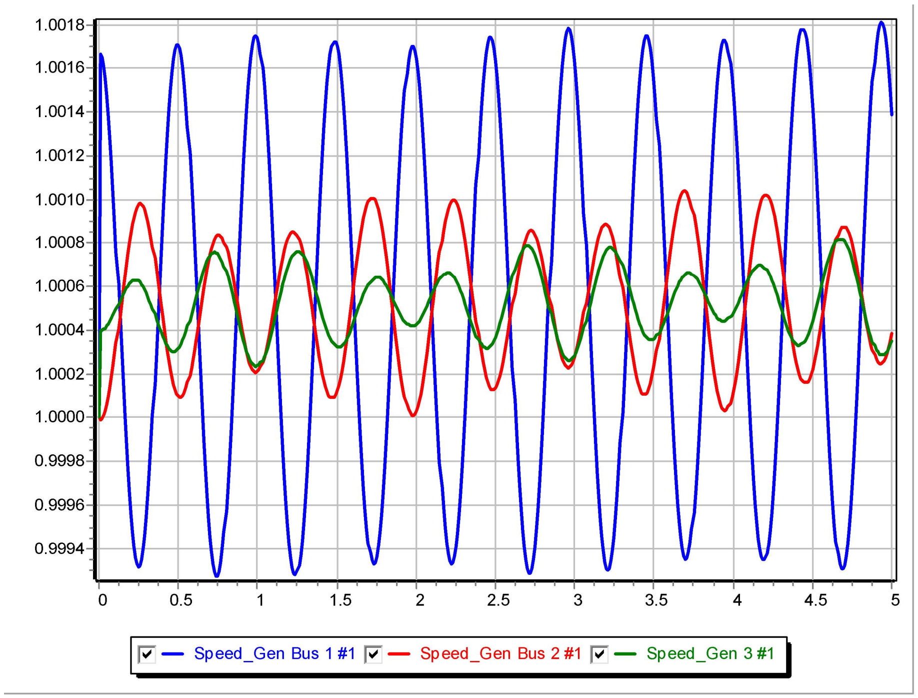
SSA Three Generator Example

- The two generator system is extended to three generators with the third generator having H_3 of 8 and X_{d3}'=0.3



SSA Three Generator Example

- Using SSA, two frequencies are identified: one at 2.02 Hz and one at 1.51 Hz



The oscillation is started with a short, self-clearing fault

Shortly we'll discuss modal analysis to determine the contribution of each mode to each signal

PowerWorld case **B2_CLS_3Gen_SSA**

Large System Studies

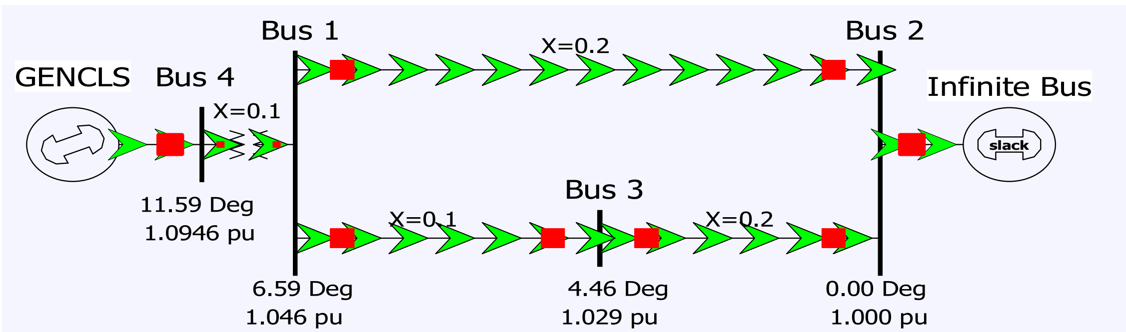
- The challenge with large systems, which could have more than 100,000 states, is the shear size
  - Most eigenvalues are associated with the local plants
  - Computing all the eigenvalues is computationally challenging, order n^3
- Specialized approaches can be used to calculate particular eigenvalues of large matrices
  - See Kundur, Section 12.8 and associated references

Single Machine Infinite Bus

- A quite useful analysis technique is to consider the small signal stability associated with a single generator connected to the rest of the system through an equivalent transmission line
- Driving point impedance looking into the system is used to calculate the equivalent line's impedance
  - The Z_{ii} value can be calculated quite quickly using sparse vector methods
- Rest of the system is assumed to be an infinite bus with its voltage set to match the generator's real and reactive power injection and voltage

Small SMIB Example

- As a small example, consider the 4 bus system shown below, in which bus 2 really is an infinite bus



Z_{44} is Z_{th} in parallel with $jX'_{d,4}$ (which is j0.3) so Z_{th} is j0.22

- To get the SMIB for bus 4, first calculate Z_{44}

$$Y_{bus} = j\begin{bmatrix} -25 & 0 & 10 & 10 \\ 0 & 1 & 0 & 0 \\ 10 & 0 & -15 & 0 \\ 10 & 0 & 0 & -13.33 \end{bmatrix} \rightarrow Z_{44} = j0.1269$$

Small SMIB Example

- The infinite bus voltage is then calculated so as to match the bus i terminal voltage and current

$$\bar{V}_{\text{inf}} = \bar{V}_i - Z_i \bar{I}_i$$

$$\text{where } \left(\frac{P_i + jQ_i}{\bar{V}_i} \right)^* = \bar{I}_i$$

While this was demonstrated on an extremely small system for clarity, the approach works the same for any size system

- In the example we have

$$\left(\frac{P_4 + jQ_4}{\bar{V}_4} \right)^* = \left(\frac{1 + j0.572}{1.072 + j0.220} \right)^* = 1 - j0.328$$

$$\bar{V}_{\text{inf}} = (1.072 + j0.220) - (j0.22)(1 - j0.328)$$

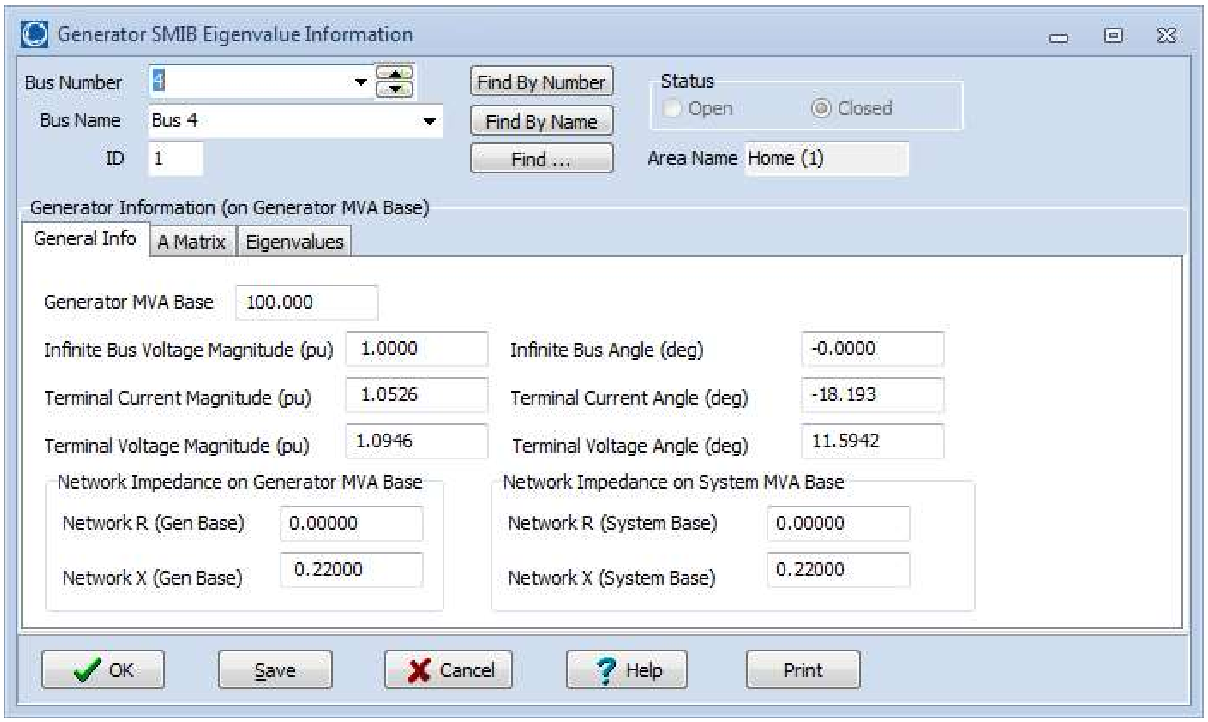
$$\bar{V}_{\text{inf}} = 1.0$$

Calculating the A Matrix

- The SMIB model **A** matrix can then be calculated either analytically or numerically
  - The equivalent line's impedance can be embedded in the generator model so the infinite bus looks like the "terminal"
- This matrix is calculated in PowerWorld by selecting Transient Stability, SMIB Eigenvalues
  - Select Run SMIB to perform an SMIB analysis for all the generators in a case
  - Right click on a generator on the SMIB form and select Show SMIB to see the Generator SMIB Eigenvalue Dialog
  - These two bus equivalent networks can also be saved, which can be quite useful for understanding the behavior of individual generators

Example: Bus 4 SMIB Dialog

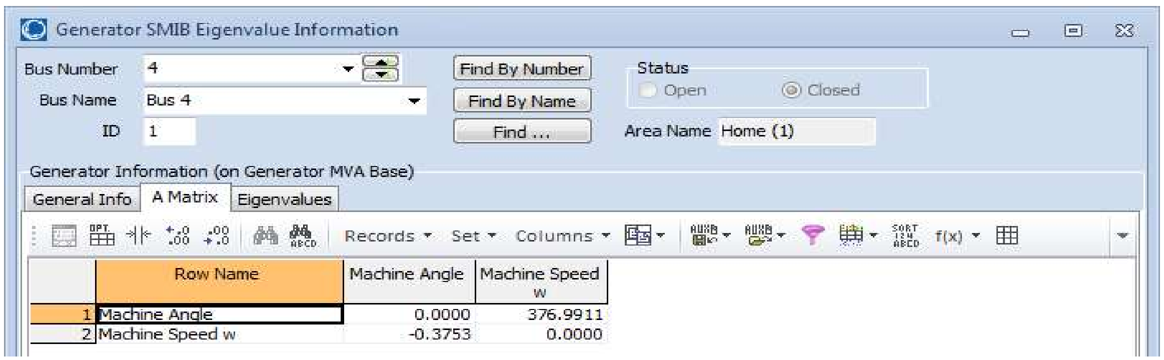
- On the SMIB dialog, the General Information tab shows information about the two bus equivalent



PowerWorld case **B4_SMIB**

Example: Bus 4 SMIB Dialog

- On the SMIB dialog, the **A** Matrix tab shows the $\mathbf{A}_{sys}$ matrix for the SMIB generator



- In this example A_{21} is showing

$$\frac{\partial \Delta\omega_{4,pu}}{\partial \delta_4} = \frac{1}{2H_4}\left(\frac{-\partial P_{E,4}}{\partial \delta_4}\right) = -\left(\frac{1}{6}\right)\left(\left(\frac{-1}{0.3+0.22}\right)\left(-1.2812\cos\left(23.94°\right)\right)\right)$$

$$= -0.3753$$

Example: Bus 4 with GENROU

- The eigenvalues can be calculated for any set of generator models
- This example replaces the bus 4 generator classical machine with a GENROU model
  - There are now six eigenvalues, with the dominate response coming from the electro-mechanical mode with a frequency of 1.84 Hz, and damping of 6.9%

Generator Information (on Generator MVA Base)

General Info | A Matrix | Eigenvalues

Records ▾ Set ▾ Columns ▾ f(x) ▾ Options ▾

	Real Part	Imag Part	Magnitude	Damping Ratio	Damped Freq (Hz)	Damped Period (Sec)	Undamped Freq (Hz)
1	-21.2472	0.0000	21.2472	1.0000	0.0000		3.3816
2	-0.8040	11.5563	11.5842	0.0694	1.8392	0.5437	1.8437
3	-0.8040	-11.5563	11.5842	0.0694	-1.8392	-0.5437	1.8437
4	-14.2256	0.0000	14.2256	1.0000	0.0000		2.2641
5	-3.7087	0.0000	3.7087	1.0000	0.0000		0.5903
6	-0.4248	0.0000	0.4248	1.0000	0.0000		0.0676

PowerWorld case **B4_GENROU_Sat_SMIB**

Example: Bus 4 with GENROU Model and Exciter

- Adding an relatively slow EXST1 exciter adds additional states (with K_A=200, T_A=0.2)
  - As the initial reactive power output of the generator is decreased, the system becomes unstable (below example is with a generator reactive power output of 0 Mvar)

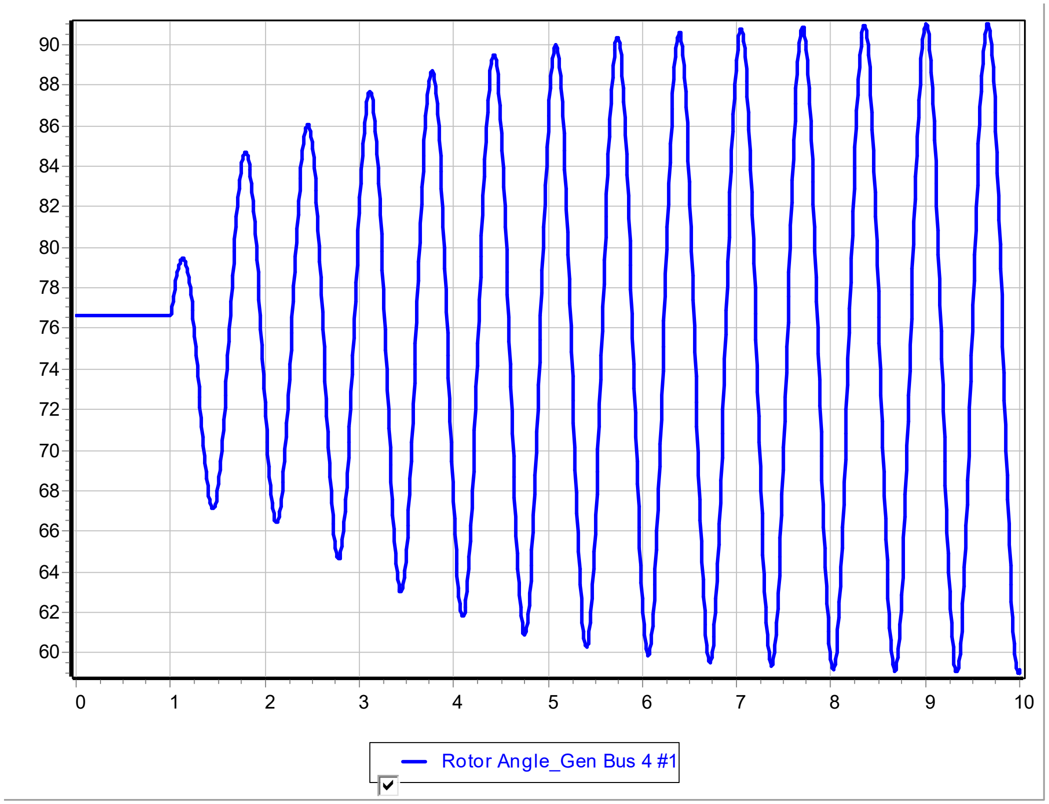
General Info | A Matrix | Eigenvalues

	Real Part	Imag Part	Magnitude	Damping Ratio	Damped Freq (Hz)	Damped Period (Sec)	Undamped Freq (Hz)	Ma
1	0.2704	-9.5336	9.5374	-0.0283	-1.5173	-0.6591	1.5179	
2	0.2704	9.5336	9.5374	-0.0283	1.5173	0.6591	1.5179	
3	-1.0000	0.0000	1.0000	1.0000	0.0000		0.1592	
4	-3.0137	0.0000	3.0137	1.0000	0.0000		0.4796	
5	-3.6849	-6.4281	7.4094	0.4973	-1.0231	-0.9775	1.1792	
6	-3.6849	6.4281	7.4094	0.4973	1.0231	0.9775	1.1792	
7	-14.4234	0.0000	14.4234	1.0000	0.0000		2.2956	
8	-21.6978	0.0000	21.6978	1.0000	0.0000		3.4533	

PowerWorld case **B4_GENROU_Sat_SMIB_QZero**

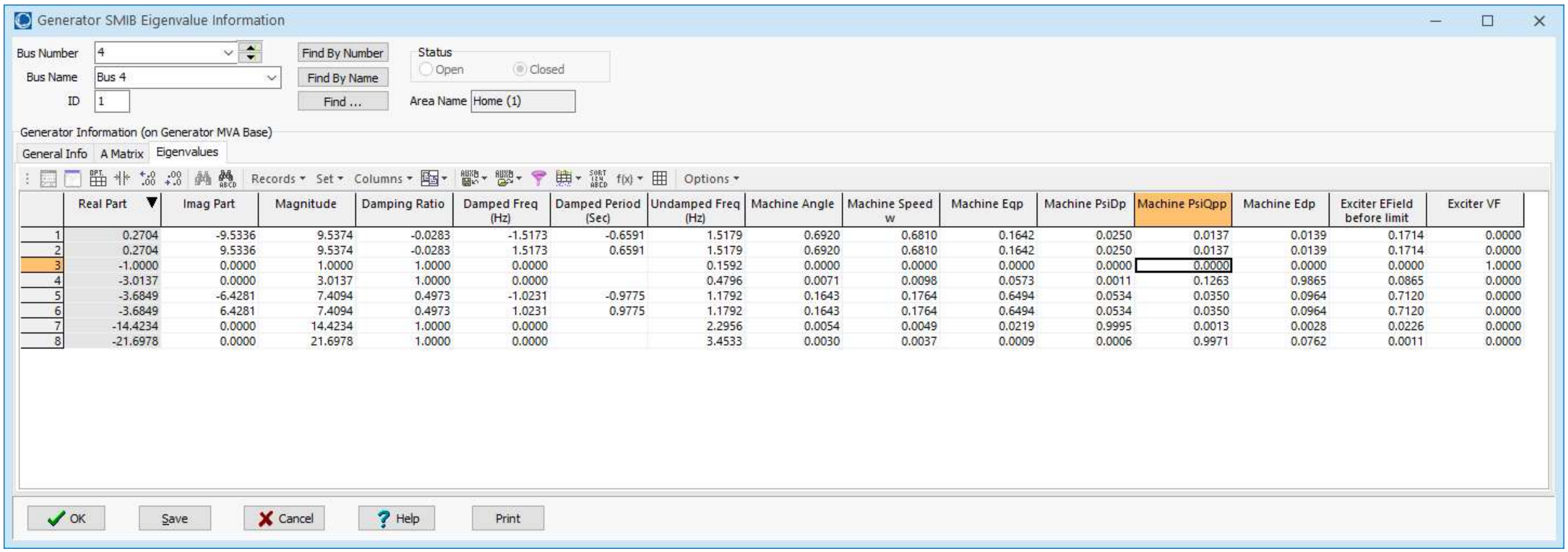
Example: Bus 4 with GENROU Model and Exciter

- The below image shows the system response to a brief bus 4 self-clearing fault



Example: Bus 4 with GENROU Model and Exciter

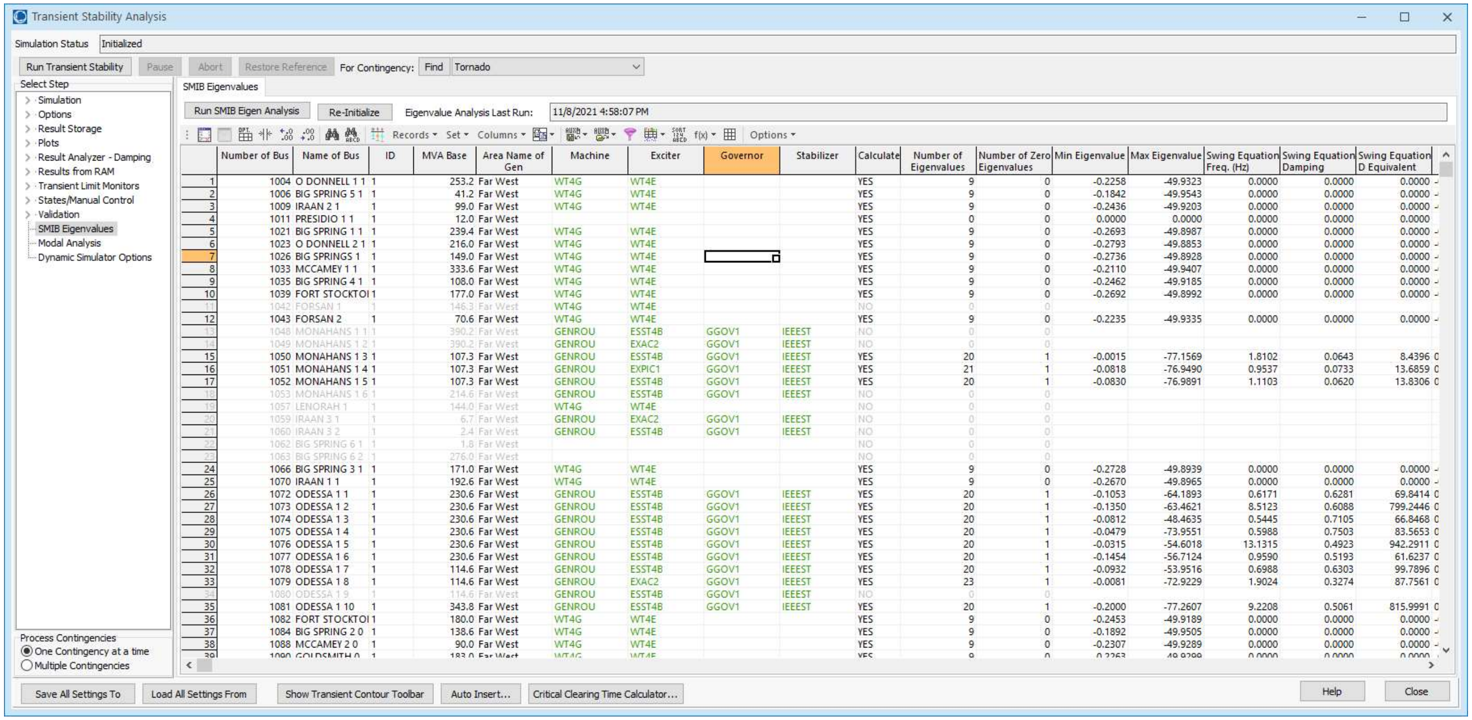
- The remainder of the Eigenvalues page shows the participation factors for the various states in the modes



	Real Part	Imag Part	Magnitude	Damping Ratio	Damped Freq (Hz)	Damped Period (Sec)	Undamped Freq (Hz)	Machine Angle	Machine Speed w	Machine Eqp	Machine PsiDp	Machine PsiQpp	Machine Edp	Exciter EField before limit	Exciter VF
1	0.2704	-9.5336	9.5374	-0.0283	-1.5173	-0.6591	1.5179	0.6920	0.6810	0.1642	0.0250	0.0137	0.0139	0.1714	0.0000
2	0.2704	9.5336	9.5374	-0.0283	1.5173	0.6591	1.5179	0.6920	0.6810	0.1642	0.0250	0.0137	0.0139	0.1714	0.0000
3	-1.0000	0.0000	1.0000	1.0000	0.0000		0.1592	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
4	-3.0137	0.0000	3.0137	1.0000	0.0000		0.4796	0.0071	0.0098	0.0573	0.0011	0.1263	0.9865	0.0865	0.0000
5	-3.6849	-6.4281	7.4094	0.4973	-1.0231	-0.9775	1.1792	0.1643	0.1764	0.6494	0.0534	0.0350	0.0964	0.7120	0.0000
6	-3.6849	6.4281	7.4094	0.4973	1.0231	0.9775	1.1792	0.1643	0.1764	0.6494	0.0534	0.0350	0.0964	0.7120	0.0000
7	-14.4234	0.0000	14.4234	1.0000	0.0000		2.2956	0.0054	0.0049	0.0219	0.9995	0.0013	0.0028	0.0226	0.0000
8	-21.6978	0.0000	21.6978	1.0000	0.0000		3.4533	0.0030	0.0037	0.0009	0.0006	0.9971	0.0762	0.0011	0.0000

SMIB Eigenvalues for TSGC_2000 Case

- All the SMIB eigenvalues can be calculated quickly even for relatively large grids



Saving a Two Bus Equivalent

- PowerWorld makes it easy to save a two bus equivalent from the **SMIB Eigenvalues** page
  - Right-click and select **Save Two Bus Equivalent**
- As the name implies, the two bus equivalent is the generator connected to an infinite bus through its driving point impedance
- Two bus equivalents provide a convenient way to track down at least some causes of instability issues

Small Signal Analysis and Measurement-Based Modal Analysis

- Small signal analysis has been used for decades to determine power system frequency response
  - It is a model-based approach that considers the properties of a power system, linearized about an operating point
- Measurement-based modal analysis determines the observed dynamic properties of a system
  - Input can either be measurements from devices (such as PMUs) or dynamic simulation results
  - The same approach can be used regardless of the measurement source
- Focus in this section is on the measurement-based approach

Ring-down Modal Analysis

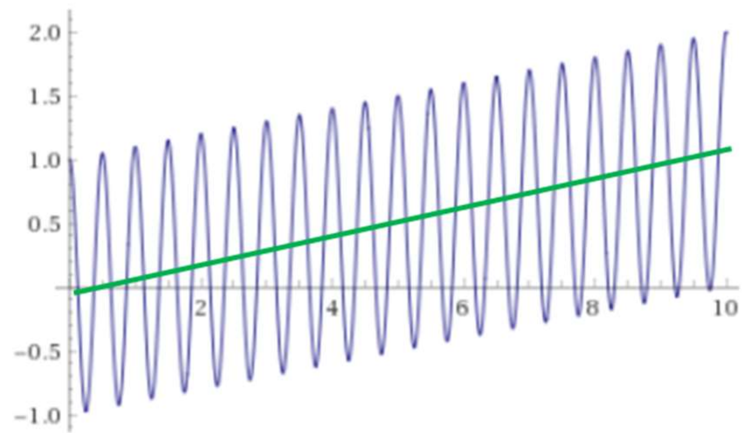
- Ring-down analysis seeks to determine the frequency and damping of key power system modes following some disturbance
- There are several different techniques, with the Prony approach the oldest (from 1795); introduced into power in 1990 by Hauer, Demeure and Scharf
- Regardless of technique, the goal is to represent the response of a sampled signal as a set of exponentially damped sinusoidals (modes)

$$y(t) = \sum_{i=1}^{q} A_i e^{\sigma_i t} \cos\left(\omega_i t + \phi_i\right) \qquad \text{Damping (\%)} = \frac{-\alpha_i}{\sqrt{\alpha_i^2 + \omega_i^2}} \times 100$$

Goal: Extracting Modes from the Signals

- The goal is to gain information about the electric grid by extracting modal information from its signals
  - The frequency and damping of the modes is key
- The premise is we'll be able to reproduce a complex signal, over a period of time, as a set a of sinusoidal modes
  - We'll also allow for linear detrending

$$0.1t + \cos(2\pi 2t)$$

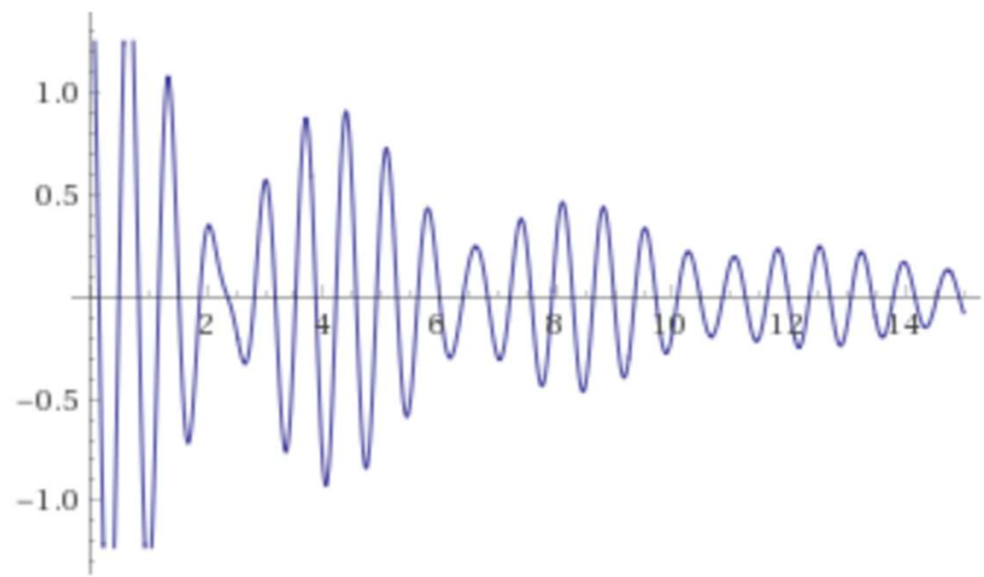


Example: Summation of Two Damped Exponentials

- This example was created by going from the modes to a signal
- We'll be going in the opposite direction (i.e., from a measured signal to the modes)

plot $e^{-0.25x}\cos(10x)+e^{-0.125x}\cos\left(8.5x+\frac{\pi}{8}\right)$

Plot:

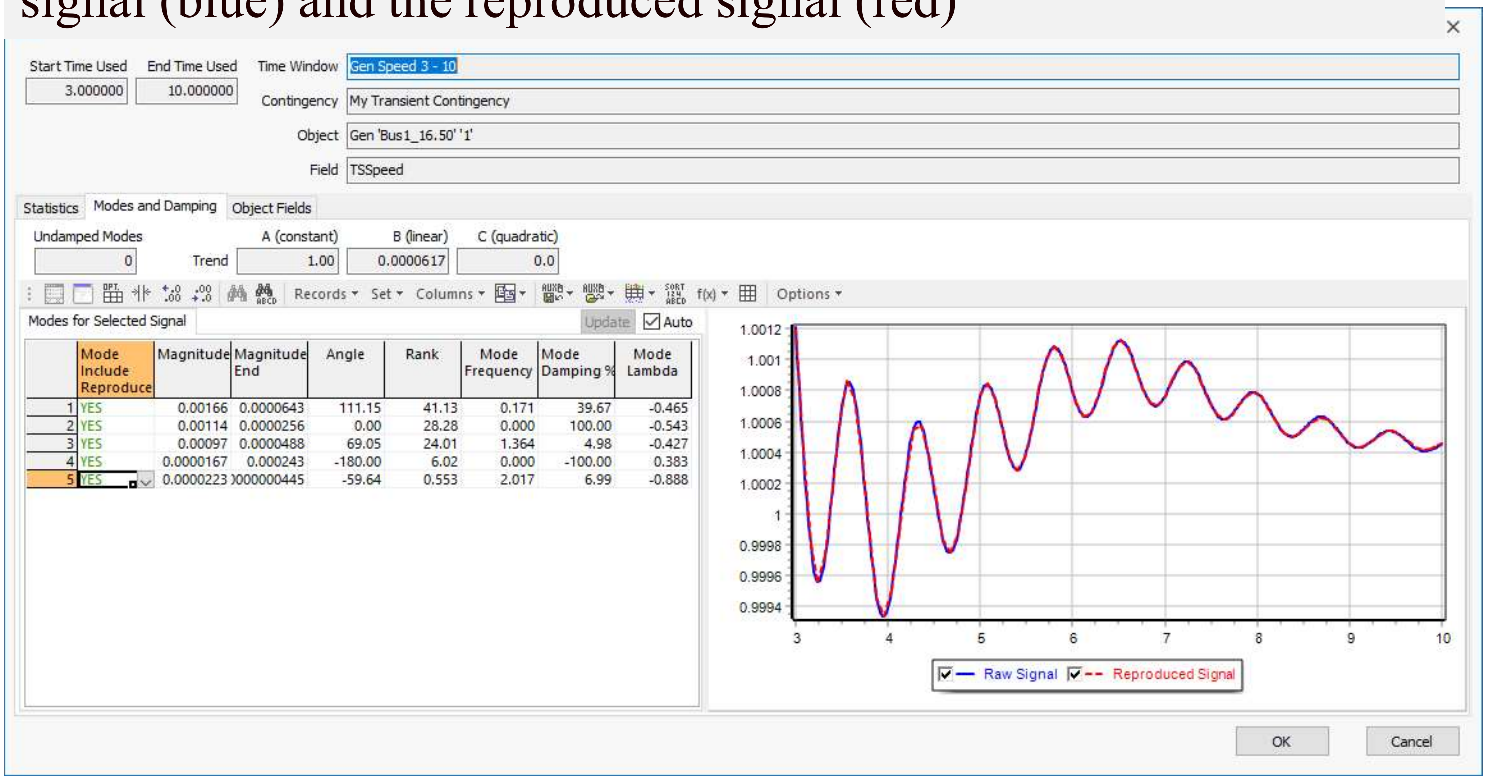


Some Reasonable Expectations

- **Verifiable** to show how well the modes match the original signal(s)
  - We'll show this
- **Flexible** to handle between one and many signals
  - We'll go up to simultaneously considering 40,000 signals
- **Fast**
  - What is presented will be, with a discussion of the computational scaling
- **Easy to use**
  - This is software implementation specific; results shown here were done using the modal analysis tool integrated into PowerWorld Simulator (version 22)

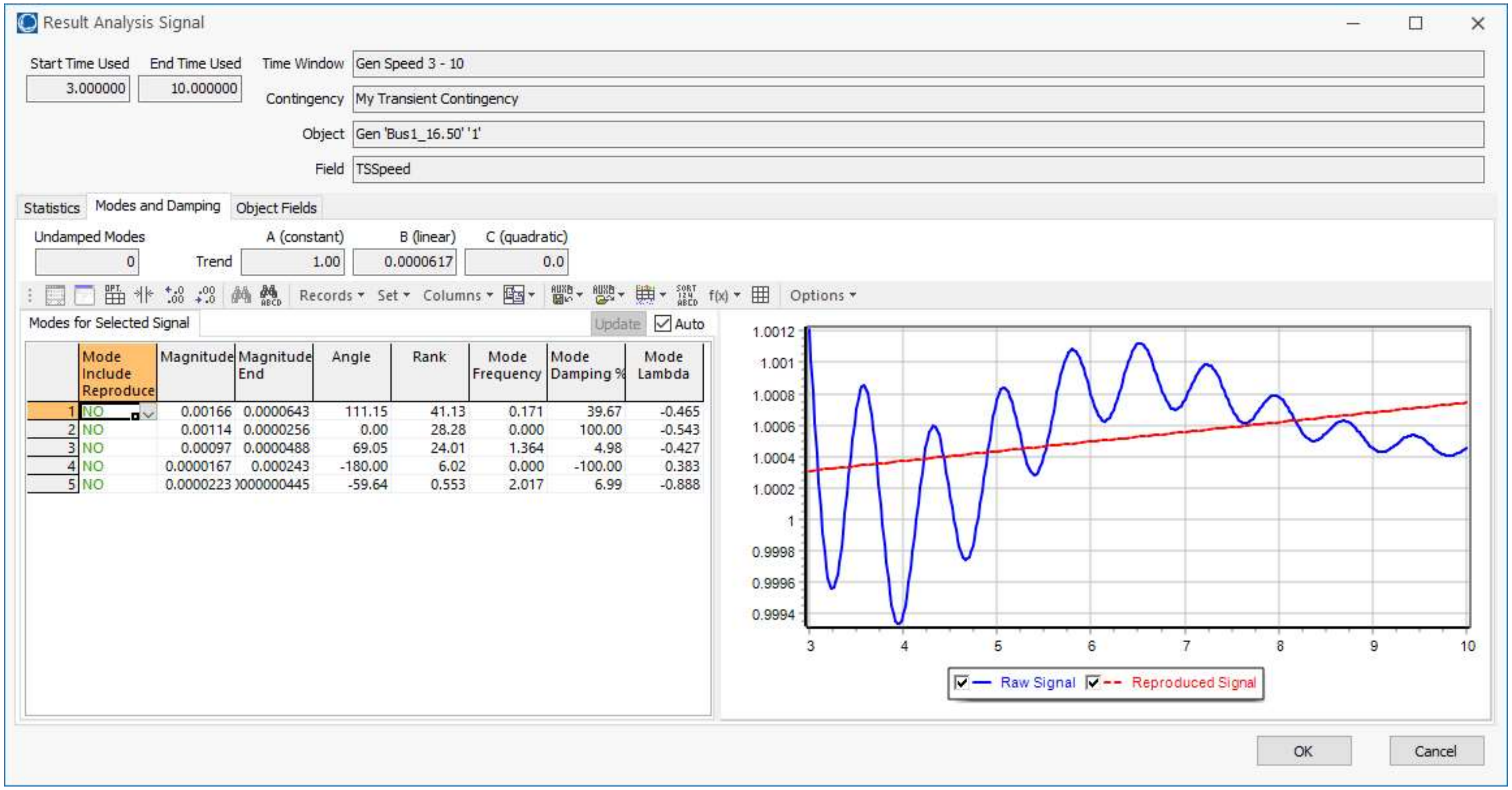
Example: One Signal

This could be any signal; image shows the result of the original signal (blue) and the reproduced signal (red)



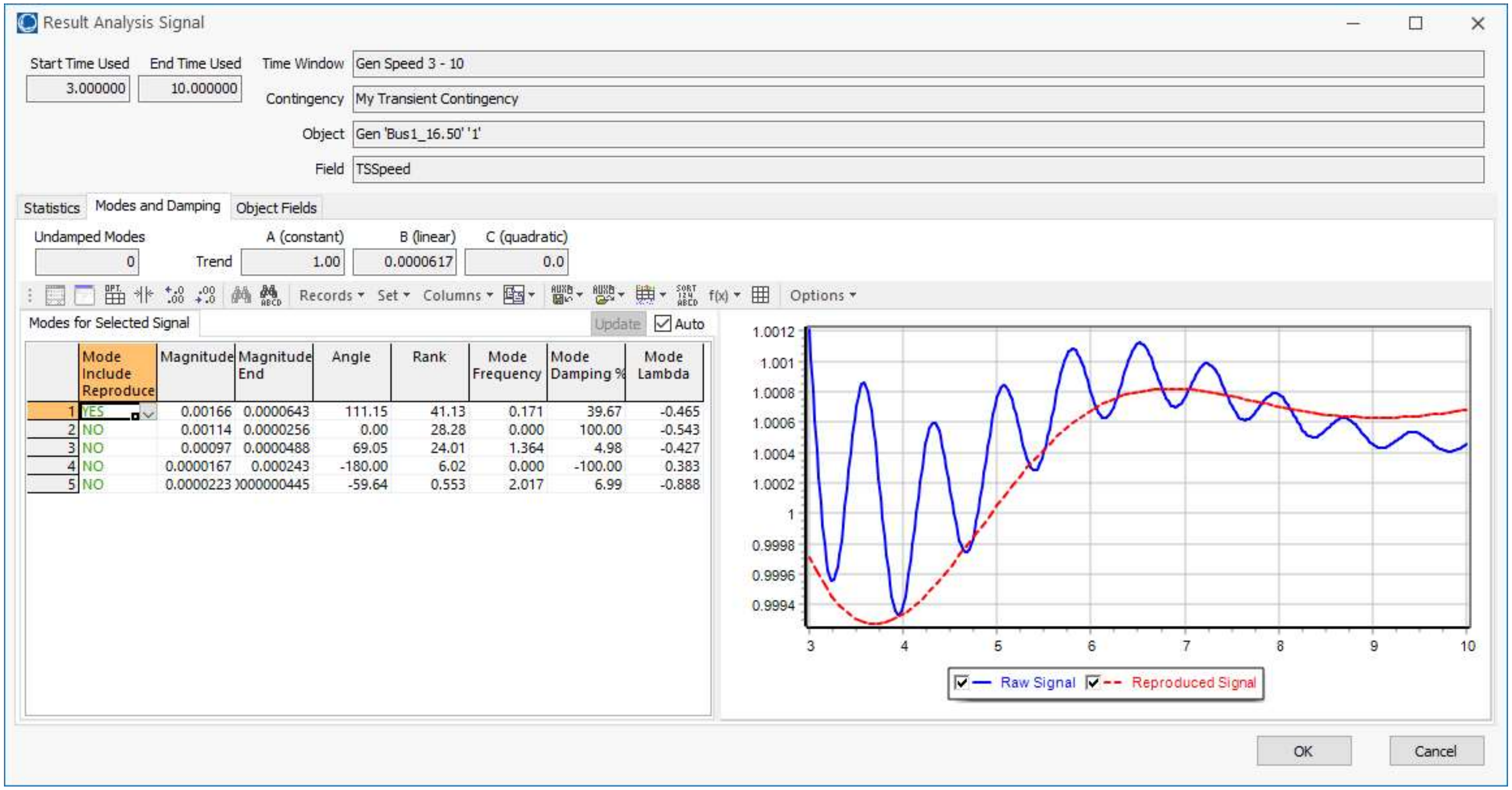
	Mode Include Reproduce	Magnitude	Magnitude End	Angle	Rank	Mode Frequency	Mode Damping %	Mode Lambda
1	YES	0.00166	0.0000643	111.15	41.13	0.171	39.67	-0.465
2	YES	0.00114	0.0000256	0.00	28.28	0.000	100.00	-0.543
3	YES	0.00097	0.0000488	69.05	24.01	1.364	4.98	-0.427
4	YES	0.0000167	0.000243	-180.00	6.02	0.000	-100.00	0.383
5	YES	0.0000223	)000000445	-59.64	0.553	2.017	6.99	-0.888

Verification: Linear Trend Line Only

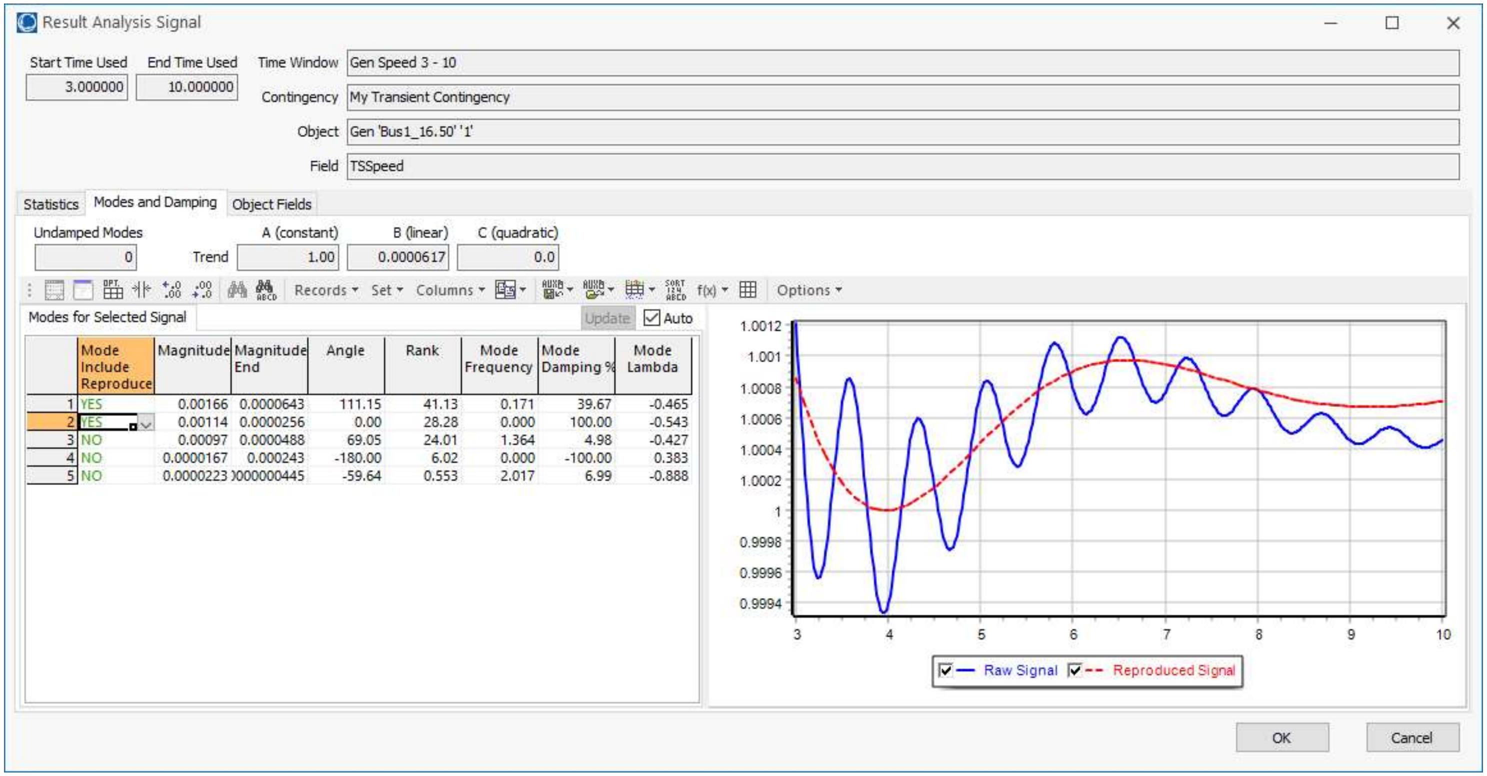


	Mode Include Reproduce	Magnitude	Magnitude End	Angle	Rank	Mode Frequency	Mode Damping %	Mode Lambda
1	NO	0.00166	0.0000643	111.15	41.13	0.171	39.67	-0.465
2	NO	0.00114	0.0000256	0.00	28.28	0.000	100.00	-0.543
3	NO	0.00097	0.0000488	69.05	24.01	1.364	4.98	-0.427
4	NO	0.0000167	0.000243	-180.00	6.02	0.000	-100.00	0.383
5	NO	0.0000223	)000000445	-59.64	0.553	2.017	6.99	-0.888

Verification: Linear Trend Line + One Mode

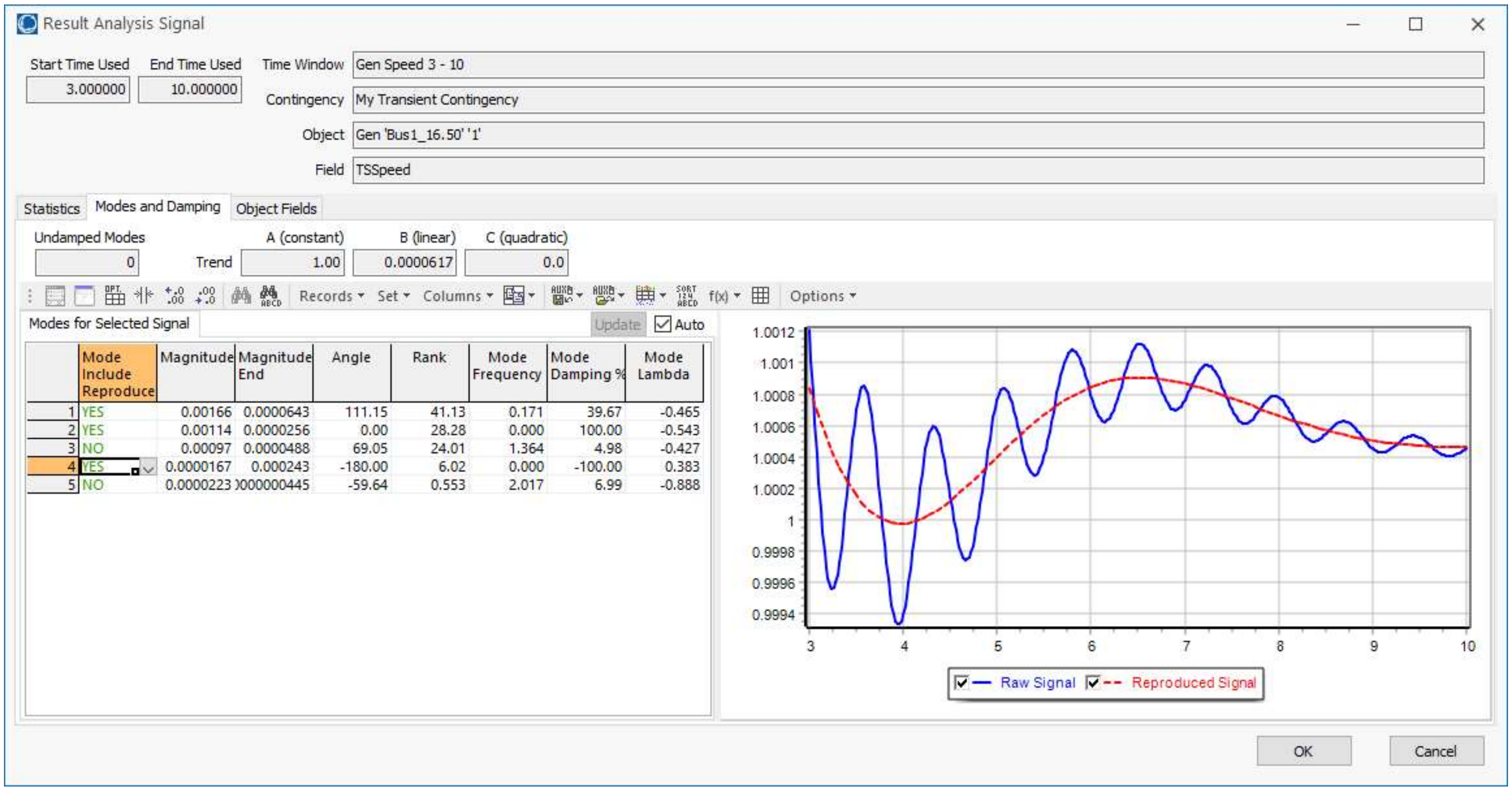


Verification: Linear Trend Line + Two Modes



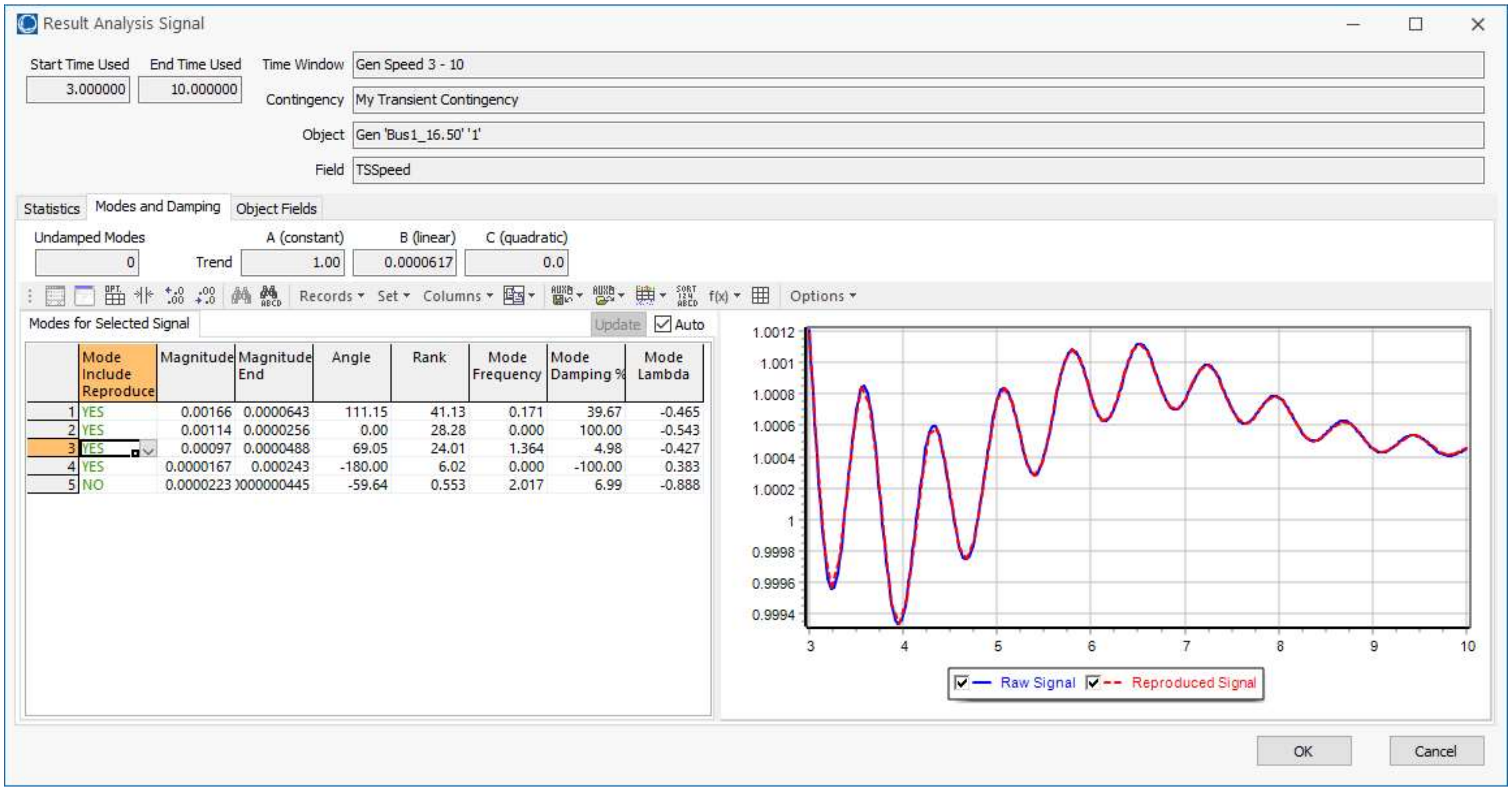
	Mode Include Reproduce	Magnitude	Magnitude End	Angle	Rank	Mode Frequency	Mode Damping %	Mode Lambda
1	YES	0.00166	0.0000643	111.15	41.13	0.171	39.67	-0.465
2	YES	0.00114	0.0000256	0.00	28.28	0.000	100.00	-0.543
3	NO	0.00097	0.0000488	69.05	24.01	1.364	4.98	-0.427
4	NO	0.0000167	0.000243	-180.00	6.02	0.000	-100.00	0.383
5	NO	0.0000223	)000000445	-59.64	0.553	2.017	6.99	-0.888

Verification: Linear Trend Line + Three Modes

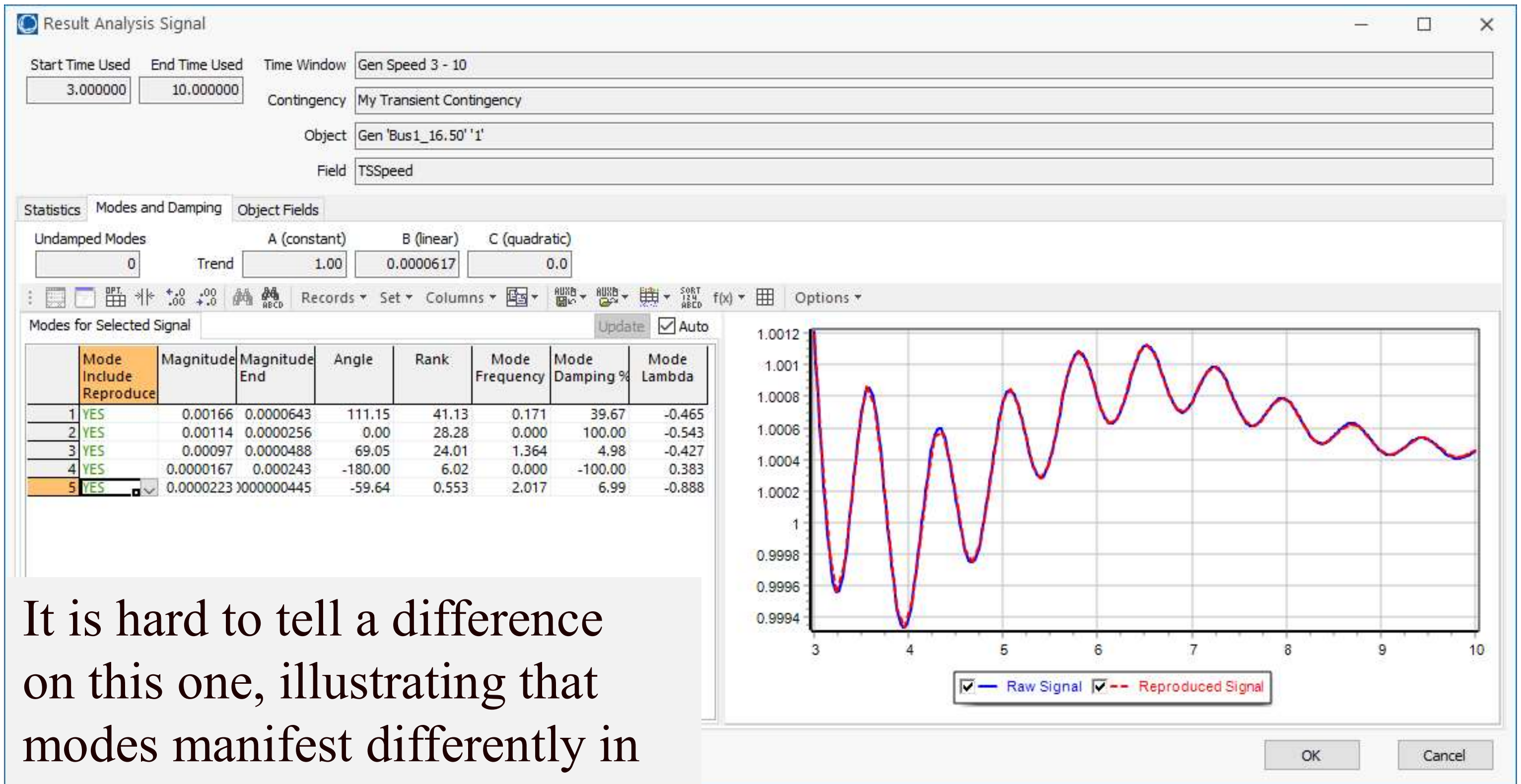


	Mode Include Reproduce	Magnitude	Magnitude End	Angle	Rank	Mode Frequency	Mode Damping %	Mode Lambda
1	YES	0.00166	0.0000643	111.15	41.13	0.171	39.67	-0.465
2	YES	0.00114	0.0000256	0.00	28.28	0.000	100.00	-0.543
3	NO	0.00097	0.0000488	69.05	24.01	1.364	4.98	-0.427
4	YES	0.0000167	0.000243	-180.00	6.02	0.000	-100.00	0.383
5	NO	0.0000223	)000000445	-59.64	0.553	2.017	6.99	-0.888

Verification: Linear Trend Line + Four Modes



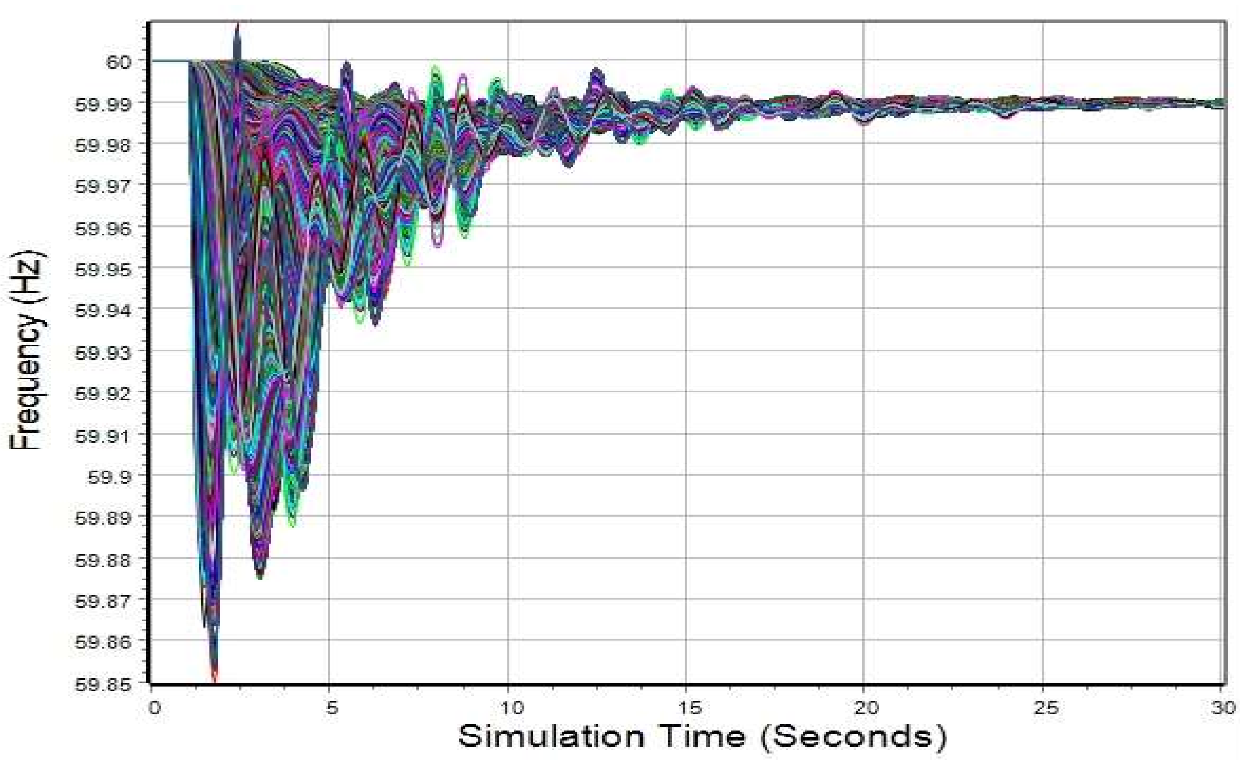
Verification: Linear Trend Line + Five Modes



It is hard to tell a difference on this one, illustrating that modes manifest differently in different signals

A Larger Example We'll Finish With

Applying the developed techniques to the response of all 43,400 substation frequencies from an 110,000 bus electric grid(20 million plus values)



Measurement-Based Modal Analysis

- There are a number of different approaches
- The idea of all techniques is to approximate a signal, $y_{org}(t)$, by the sum of other, simpler signals (basis functions)
  - Basis functions are usually exponentials, with linear and quadratic functions used to detrend the signal
  - Properties of the original signal can be quantified from basis function properties
    - Examples are frequency and damping
  - Signal is considered over time with t=0 as the start
- Approaches sample the original signal $y_{org}(t)$

Measurement-Based Modal Analysis

- Vector **y** consists of m uniformly sampled points from $y_{org}(t)$ at a sampling value of ΔT, starting with t=0, with values y_j for j=1…m
  - Times are then $t_j = (j-1)\Delta T$
  - At each time point j, the approximation of y_j is

$$\hat{y}_j(\mathrm{t}_j, \boldsymbol{\alpha}) = \sum_{i=1}^{n} b_i \phi_i(t_j, \boldsymbol{\alpha})$$

where $\boldsymbol{\alpha}$ is a vector with the real and imaginary eigenvalue components,

with $\phi_i(t_j, \boldsymbol{\alpha}) = e^{\alpha_i t_j}$ for α_{i} corresponding to a real eigenvalue, and

$\phi_i(t_j, \boldsymbol{\alpha}) = e^{\alpha_i t_j} \cos(\alpha_{\mathrm{i+1}} \mathrm{t}_j)$ and $\phi_{i+1}(\boldsymbol{\alpha}) = e^{\alpha_i t_j} \sin(\alpha_{\mathrm{i+1}} \mathrm{t}_j)$

for a complex eigenvector value

Measurement-Based Modal Analysis

- Error (residual) value at each point j is

$$r_j(t_j, \boldsymbol{\alpha}) = y_j - \hat{y}_j(t_j, \boldsymbol{\alpha})$$

- The closeness of the fit can be quantified using the Euclidean norm of the residuals

$$\frac{1}{2}\sum_{j=1}^{m}(y_j - \hat{y}_j(t_j, \boldsymbol{\alpha}))^2 = \frac{1}{2}\left\|\mathbf{r}(\boldsymbol{\alpha})\right\|_2^2$$

- Hence we need to determine α and **b**

$$\hat{y}_j(\mathrm{t}_j, \boldsymbol{\alpha}) = \sum_{i=1}^{n} b_i \phi_i(t_j, \boldsymbol{\alpha})$$

Sampling Rate and Aliasing

- The Nyquist-Shannon sampling theory requires sampling at twice the highest desired frequency
  - For example, to see a 5 Hz frequency we need to sample the signal at a rate of at least 10 Hz
- Sampling shifts the frequency spectrum by 1/T (where T is the sample time), which causes frequency overlap
- This is known as aliasing, which can cause a high frequency signal to appear to be a lower frequency signal
  - Aliasing can be reduced by fast sampling and/or low pass filters

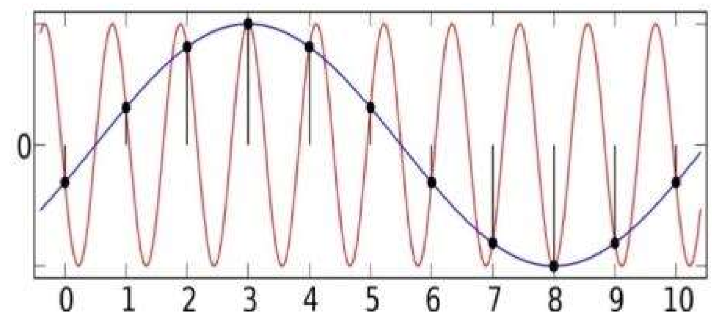


Image: upload.wikimedia.org/wikipedia/commons/thumb/2/28/AliasingSines.svg/2000px-AliasingSines.svg.png

One Solution Approach: The Matrix Pencil Method

- There are several algorithms for finding the modes. We'll use the Matrix Pencil Method
  - This is a newer technique for determining modes from noisy signals (from about 1990, introduced to power system problems in 2005); it is an alternative to the Prony Method
  - The Matrix Pencil Method is useful when there is signal noise
- Given m samples, with L=m/2, the first step is to form the Hankel Matrix, **Y** such that

This not a sparse matrix

$$\mathbf{Y} = \begin{bmatrix} y_1 & y_2 & \cdots & y_{L+1} \\ y_2 & y_3 & \cdots & y_{L+2} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m-L} & y_{m-L+1} & \cdots & y_m \end{bmatrix}$$

Reference: A. Singh and M. Crow, "The Matrix Pencil for Power System Modal Extraction," IEEE Transactions on Power Systems, vol. 20, no. 1, pp. 501-502, Institute of Electrical and Electronics Engineers (IEEE), Feb 2005.

Algorithm Details, cont.

- Then calculate **Y**'s singular values using an economy singular value decomposition (SVD)

$$\mathbf{Y} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathbf{T}}$$

- The ratio of each singular value is then compared to the largest singular value σ_c; retain the ones with a ratio > than a threshold
  - This determines the modal order, M
  - Assuming **V** is ordered by singular values (highest to lowest), let $\mathbf{V}_p$ be then matrix with the first M columns of **V**

The computational complexity increases with the cube of the number of measurements!

This threshold is a value that can be changed; decrease it to get more modes.

Aside: The Matrix Singular Value Decomposition (SVD)

- The SVD is a factorization of a matrix that generalizes the eigendecomposition to any m by n matrix to produce

$$\mathbf{Y} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathbf{T}}$$

The original concept is more than 100 years old, but has found lots of recent applications

where Σ is a diagonal matrix of the singular values

- The singular values are non-negative, real numbers that can be used to indicate the major components of a matrix (the gist is they provide a way to decrease the rank of a matrix)

Aside: SVD Image Compression Example

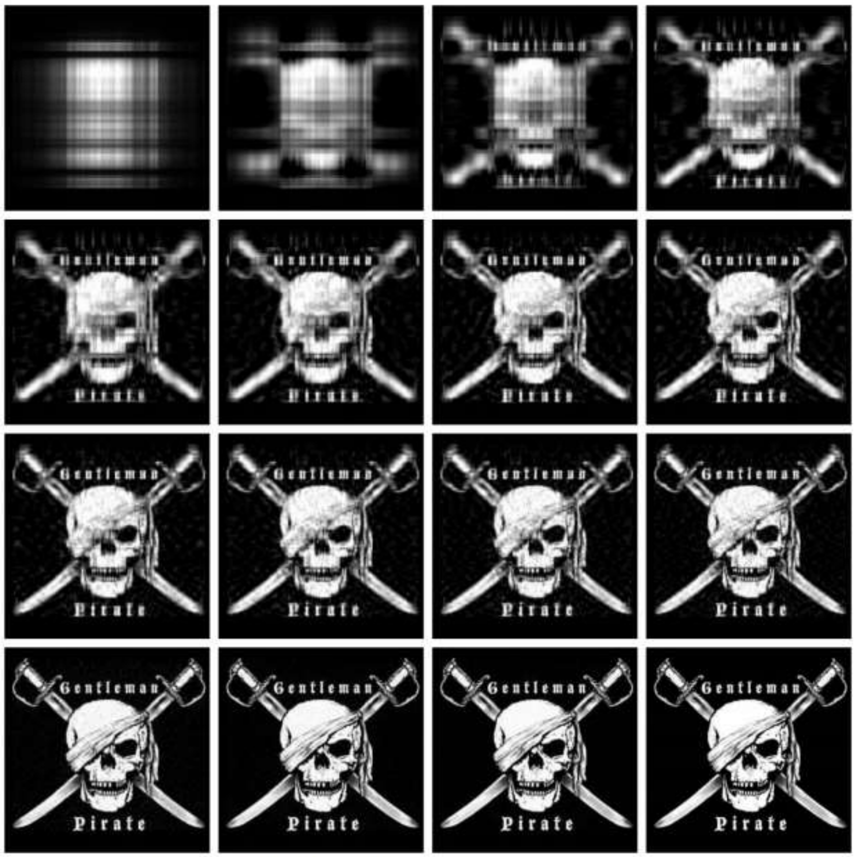


Figure 3.1: Image size 250x236 – modes used {{1,2,4,6},{8,10,12,14},{16,18,20,25},{50,75,100,original image}}

Images can be represented with matrices. When an SVD is applied and only the largest singular values are retained the image is compressed.

Image Source: www.math.utah.edu/~goller/F15_M2270/BradyMathews_SVDImage.pdf

Matrix Pencil Algorithm Details, cont.

- Then form the matrices $\mathbf{V}_1$ and $\mathbf{V}_2$ such that
  - $\mathbf{V}_1$ is the matrix consisting of all but the last row of $\mathbf{V}_p$
  - $\mathbf{V}_2$ is the matrix consisting of all but the first row of $\mathbf{V}_p$
- Discrete-time poles are found as the generalized eigenvalues of the pair $(\mathbf{V}_2^T\mathbf{V}_1, \mathbf{V}_1^T\mathbf{V}_1) = (\mathbf{A},\mathbf{B})$
- These eigenvalues are the discrete-time poles, z_i with the modal eigenvalues then

If **B** is nonsingular (the situation here) then the generalized eigenvalues are the eigenvalues of $\mathbf{B}^{-1}\mathbf{A}$

$$\lambda_i = \frac{\ln(z_i)}{\Delta T}$$

The log of a complex number $z=r\angle\theta$ is $\ln(r) + j\theta$

Matrix Pencil Method with Many Signals

- The Matrix Pencil approach can be used with one signal or with multiple signals
- Multiple signals are handled by forming a $\mathbf{Y}_{\mathrm{k}}$ matrix for each signal k using the measurements for that signal and then combining the matrices

$$\mathbf{Y}_k = \begin{bmatrix} y_{1,\mathrm{k}} & y_{2,\mathrm{k}} & \cdots & y_{L+1,\mathrm{k}} \\ y_{2,\mathrm{k}} & y_{3,k} & \cdots & y_{L+2,\mathrm{k}} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m-L,\mathrm{k}} & y_{m-L+1,\mathrm{k}} & \cdots & y_{m,\mathrm{k}} \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 \\ \vdots \\ \mathbf{Y}_N \end{bmatrix}$$

The required computation scales linearly with the number of signals

Matrix Pencil Method with Many Signals

- However, when dealing with many signals, usually the signals are somewhat correlated, so vary few of the signals are actually need to be included to determine the desired modes
- Ultimately we are finding

$$y_j(\mathrm{t}_j, \boldsymbol{\alpha}) = \sum_{i=1}^{n} b_i \phi_i(t_j, \boldsymbol{\alpha})$$

- The α is common to all the signals (i.e., the system modes) while the **b** vector is signal specific (i.e., how the modes manifest in that signal)

Quickly Determining the b Vectors

- A key insight is from an approach known as the Variable Projection Method (from Borden, 2013) that for any signal k

$$\mathbf{y}_k = \mathbf{\Phi}(\boldsymbol{\alpha})\mathbf{b}_k$$

And then the residual is minimized by selecting $\mathbf{b}_k = \mathbf{\Phi}(\boldsymbol{\alpha})^+ \mathbf{y}_k$

where $\mathbf{\Phi}(\boldsymbol{\alpha})$ is the m by n matrix with values

$\Phi_{ji}(\boldsymbol{\alpha}) = e^{\alpha_i t_j}$ if α_i corresponds to a real eigenvalue,

and $\Phi_{ji}(\boldsymbol{\alpha}) = e^{\alpha_i t_j} \cos(\alpha_{i+1} t_j)$ and $\Phi_{ji+1}(\boldsymbol{\alpha}) = e^{\alpha_i t_j} \sin(\alpha_{i+1} t_j)$

for a complex eigenvalue; $t_j = \left(j-1\right)\Delta T$

Finally, $\mathbf{\Phi}(\boldsymbol{\alpha})^+$ is the pseudoinverse of $\mathbf{\Phi}(\boldsymbol{\alpha})$

Where m is the number of measurements and n is the number of modes

A. Borden, B.C. Lesieutre, J. Gronquist, "Power System Modal Analysis Tool Developed for Industry Use," *Proc. 2013 North American Power Symposium*, Manhattan, KS, Sept. 2013