Integrating Voltage Security Measures: Connections Between Steady State, Small Disturbance Stability, and Nonlinear Dynamic Criteria

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ABSTRACT

Research literature examining the phenomena of voltage collapse has increasing converged towards an explanation involving quasi-static evolution of the system operating point towards a condition where that operating point loses asymptotic stability. However, many works have also recognized the need to supplement such studies with information reflecting the nonlinear dynamics of the system. This paper will review a range of existing results on voltage collapse security measures, and will show a range of power system models where the relationship between quasi-static and dynamic security measures can be clearly established. It will be shown that the similarities and differences of the resulting security measures can best be understood by examining the time scale of the external load and parameter variations that drive the evolution of the power system.

I. Introduction and Background

The threat of voltage collapse and voltage instability phenomena has attracted considerable attention in the power system community over the past decade, as indicated by a tremendous volume of research literature that has been produced on the topic. With the accumulating experience of the power system community, there is growing recognition of the various aspects of the problem, combining as it does both slow and fast time scale phenomena, and quasi-static analysis methods with dynamic models. This paper will contrast various methods of analysis in a consistent system model.

One of the main goals of voltage stability analysis for power system operation is the identification of a useful, computationally tractable security index that can be used to judged a system's vulnerability to voltage collapse, and to evaluate the effectiveness of various corrective strategies when the security of the system is threatened. Different modelling assumptions, and the mix of static and dynamic criteria has often made it difficult to effectively compare various voltage security measure. One of the most comprehensive attempts at such comparison is the work reported in [1], where various measures are compared in terms of a common criteria based on statistical decision theory. In that work various methods are compared in terms of their ability to reliably classify operating conditions as secure or insecure with respect to voltage collapse. Among the conclusions of that work was the observation that "one needs at least one static and one dynamic criterion to identify the voltage collapse problem," implying that there is fundamentally different information obtained from the two classes of indices. However, as the understanding of voltage collapse and the related system modelling required to represent the problem has evolved, it has become clear that there are close connections between static and dynamic criteria. This paper will attempt to further illustrate these connections, and will argue that for a wide class of models the information obtained from each type of measure can be self consistent (predicting the same ultimate operational limits to load and parameter changes), and complementary (with quasi-static measures naturally capturing slower time scale parameter variations, dynamic measures capturing the effect of faster time scale parameter variations) Many researchers have previously addressed connections between static and dynamic approaches to voltage collapse studies, so little in this paper will be truly new, but the authors hope that collection of a range of results in a single consistent presentation (as well as an updated interpretation of the authors' own work) will prove useful. This paper is not, however, intended as a comprehensive review. The range of papers on voltage collapse is so extensive that it is unfortunately inevitable that many relevant works will not be adequately discussed here. To the researchers responsible for these works, the authors offer their apologies. The field will await a comprehensive survey at some future date.

This paper will be organized as follows. A general structure of equations common to power system models will be discussed first, and the role of both slow and fast time scale parameter and load variations in these models will be highlighted. With this structure established, three major "schools of thought" regarding voltage collapse analysis will be reviewed. These will include bifurcation analyses, sensitivity analyses, and nonlinear dynamic analyses based on energy functions. For the last of these approaches, a new interpretation in terms of an optimal control problem will be introduced, which clarifies the role of fast time scale disturbances in this analysis. The discussion of nonlinear dynamic measures will also draw connections to voltage collapse proximity indicators which make use of multiple power flow solutions, which play an important role in the optimal control problem. The final section of the paper will then restrict attention to a specific power system model (though *not* specific network data), and will show that in this model the various analysis methods offer results that are very closely related, with differences arising only from the time scale of parameter or load variation assumed.

II. General Modeling for Integrated Analysis

To facilitate an examination of both quasi-static bifurcation and dynamic analyses, as well as more classical sensitivity analyses, it is useful to look first at a general structure of equations that encompasses the nonlinear dynamic model. Working just with the general structure of equations, it is possible to illustrate many of the necessary assumptions that underlie sensitivity and bifurcation analyses, and to contrast these with Lyapunov based nonlinear stability analyses. However, a general abstract description of equation structure is not sufficient to examine all of the practical modeling issues that influence voltage stability. Therefore, following the general structure of equations to be presented below, a specific power system model that captures many of the characteristics relevant to voltage collapse will be constructed in Section VI.

The most commonly used structure of equations used in power system dynamic simulation programs is that of differential equations with algebraic constraints [2]. For voltage collapse analysis, it will be useful to "pull out" the dependence of these equations on time varying load parameters, which will be denoted here as $L(t)$. In simple constant P-Q load models, these parameters will simply be variations in active and reactive load. For voltage dependent loads (referred to as "nonlinear loads" by some authors [3]), these might be time varying coefficients in a polynomial, reflecting a time varying change in the composition of a particular load. In this later case, these time varying coefficients may multiply system variables such as voltage, so the vector $L(t)$ may be premultiplied by a state dependent matrix. The resulting system model will be have the following structure.

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, \mathbf{y}) + \mathbf{B}^1(\mathbf{z}, \mathbf{y})\mathbf{L}(t) \tag{1a}$$

$$\mathbf{0} = \mathbf{g}(\mathbf{z}, \mathbf{y}) + \mathbf{B}^2(\mathbf{z}, \mathbf{y})\mathbf{L}(t) \tag{1b}$$

For example, consider a very simple case of this general structure: a single generator, represented by a classical model, feeding a load that is a sum of a constant P-Q term and a time varying inductive term. The dynamic vector $\mathbf{z}$ would have two components, corresponding to the generator angle and frequency; the single algebraic variable y would correspond to the load bus voltage magnitude. The model would have two first order differential equations, defining derivative of frequency and angle, and a single algebraic constraint imposing reactive power balance at the load bus. The reactive power balance equation would contain the single time varying $L(t)$ term, representing the varying inductive load, and this term would be premultipled by voltage magnitude squared (i.e., $\mathbf{B}^2(\mathbf{z},y) = y^2$) .

As noted above, this type of mixed system of differential equations with algebraic constraints is very commonly used in power systems simulations. However, the behavior of trajectories in such systems can be markedly different from that of pure systems of differential equations alone. For example, it is possible for trajectories to exist over some interval of time, but disappear at some later point in the trajectory. This is phenomena has been labelled "impasse point" behavior in circuit simulations [4], [5], where it usually indicates that the model is deficient because important parasitic dynamics have been neglected. Also, when comparing two systems that have the same steady state equations for equilibria, the stability of equilibrium points can change wildly depending on whether or not some the defining equations remain as algebraic constraints in the full dynamic model. Not surprisingly, techniques for estimating regions of attraction must also be modified to accommodate such mixed differential/algebraic systems. These issues have begun to be addressed in the power systems literature in such works as [6] and [7], but deserve further scrutiny in voltage stability analyses. This point will be raised again in reviewing bifurcation analyses.

To resolve the difficulties inherent in differential/algebraic representations one may seek physically plausible *dynamic* models for load that admit the same equilibria as predicted by the algebraic load model. One approach to introducing such dynamics is reported in [8], using a load model described in [9]. In these works, the steady state load model used in powerflow analysis is modified. An alternative approach that keeps the standard steady state powerflow model is described in [7], where the reactive power balance equations that force a system to always have zero mismatch at each bus are "perturbed" to a dynamic model that allows instantaneous mismatch to exist. The new dynamics are assumed to have a fast time constant that moves voltage magnitude back to an equilibrium with zero mismatch. This involves modifying the reactive power mismatch equations by introducing a term dependent on the derivative of bus voltage magnitude. With either of these approaches, the result is a new system model composed purely a state space model, much like the result of adding of parasitic dynamics in the circuit case of [5]. In this case, (1) is condensed to the form:

$$\dot{x} = f(x) + B(x)L(t) . \qquad (2)$$

A first step in comparing various analysis techniques lies in identifying the assumed time scales of the parameter variations that enter models such as (1) and (2). Load studies in power systems have shown that aggregate variation in demand at a distribution substation level can often be decomposed into two major components: a relatively large magnitude "moving average" component, slowly varying on a time scale of minutes to hours, and a smaller magnitude, zero mean component, varying on a time scale of seconds [10], [11]. This separation of time scales is represented by decomposing the L(t) term into a slow time scale component, and a small magnitude, zero mean component, as indicated below.

$$L(t) = L_{slow}(t) + L_{small}(t) \qquad (3)$$

The first component of (3) captures the familiar hourly variation in the "load curve" through the course of 24 hours or a week, and is used to determine loading levels in most types of power system studies. The second component models the inherently random behavior that results from aggregating thousands of individual switching actions in customer loads. Once the slowly varying average component is removed, this remaining fast time scale variation is typically quite small in magnitude (a few percent of nominal load), and is often considered negligible in standard power system stability studies. However, operating conditions associated with voltage collapse typically display very high sensitivity to small variations in load. Hence, later in this paper we will argue that it may be advisable to keep even such small variations explicitly represented in the model. However, as a first step, consider the effect of including only the term $L_{slow}(t)$ from (3) in (2), and assume that the time rate of change of this term is indeed slow relative to the time constants of the dynamics. Intuitively, one expects that the evolution of the system state, x(t), will tend to be close to the trajectory, $\hat{x}(t)$, predicted by solving the quasi-static equilibrium equation below:

$$0 = f(\hat{x}(t)) + B(\hat{x}(t))L_{slow}(t) . \qquad (4)$$

This should remain valid provided that the dynamics of (2), when linearized about $\hat{x}(t)$, are stable with time constants much faster that the variation of $\hat{x}(t)$. Rigorous conditions for $\|x(t) - \hat{x}(t)\|$ to remain bounded by a given constant are easily obtained in the case when the system has a suitable Lyapunov function, and are derived in [12]. Voltage collapse analysis may be viewed as an attempt to efficiently predict (or control against) operating conditions that ultimately fail to satisfy these conditions, since collapse is associated with x(t) (particularly bus voltages) diverging from the desired operating point predicted by $\hat{x}(t)$.

III. Bifurcation Analysis

The discussion of this section will assume that small magnitude random variations in load are negligible, and that $L_{slow}(t)$ defines the only parameter variation of interest. For simplicity of notation in this section, this load variation will be denoted only as L(t). One expects that trajectories x(t) satisfying (2) will diverge if (4) ceases to have a solution altogether, or if $\hat{x}(\hat{t})$ no longer defines a (sufficiently) stable equilibrium for (2) when the load term is "frozen" at $L(\hat{t})$. This is the viewpoint adopted in voltage stability bifurcation analyses such as [13] and [14]. These approaches typically assume that a suitable load prediction exists to define the function L(t), with a starting value L(0) that yields an acceptable, stable operating point. The goal then becomes one of determining if there exists a time $\hat{t}$, or equivalently, a load level $L(\hat{t})$, at which the assumed operating point $\hat{x}(\hat{t})$ looses small disturbance stability. For such bifurcation analyses, stability of the operating point is judged by examining the linearization of the dynamics about the points predicted by $\hat{x}(t)$. However, considerable information regarding the *local* behavior of the true nonlinear trajectories can be obtained from such linearized analysis. For example, [14] shows that when the true state x(t) diverges from $\hat{x}(t)$, its initial direction of motion can be predicted by an eigenvector obtained from the linearization about the critical value of $\hat{x}(\hat{t})$.

Identification of critical loading levels $L(\hat{t})$ is facilitated by the fact that for many simplified dynamic models, the critical loading is associated with an operating point where the power flow Jacobian, denoted by J(x), is singular. Relating this observation to (2), this property arises if the linearization $\partial f/\partial x$ has a zero eigenvalue when J(x) (which often makes up a sub-block of $\partial f/\partial x$) is singular. Section VI will examine a specific class of system models relevant to voltage collapse for which this property holds. The critical role of power flow Jacobian has been recognized for many years, and motivates a number of security measures that measure proximity to voltage collapse by the distance of the power flow Jacobian from singularity [15]. From a computational standpoint, so long as the load variation is function of one parameter (such as time t in the L(t) model here), it is a straightforward matter to augment steady state power flow equations with constraints that force the power flow Jacobian to be singular. In this way, the critical parameter value $\hat{t}$ can be solved with very little more computation than a standard power flow. This general computational approach is described in [16], with the power systems application developed in [17]. In many models, the critical value $\hat{t}$ typically predicts a point where two power flow solutions coalesce, and for further increase in t, the powerflow no longer has a solution in the neighborhood of $\hat{x}(\hat{t})$.

For more detailed dynamic power system models, the linearized dynamic about an operating point predicted by $\hat{x}(t)$ may loose stability when complex eigenvalues cross the imaginary axis, rather than by a real eigenvalue crossing through zero. In the former case, one expects the physical instability to be manifest by growing oscillations[1]; in the later, the divergence of state would be an exponential with positive real exponent. Hopf bifurcation associated with imaginary axis eigenvalue crossings was studied in [18], where it was shown that variation in exciter gains could yield this type of transition to oscillatory instability. More recent work in [19] has suggested that oscillations associated with generator exciter and field dynamics may be responsible for some types of voltage instability. However, the field reports on voltage collapse incidents [20], [21] suggest that the divergence of voltage magnitudes from the operating point displayed a monotonic decline, rather than growing sinusoidal oscillation. This admittedly limited evidence would seem to indicate that the instability of voltage collapse should be modelled by a purely real eigenvalue crossing through the origin.

To summarize, the general bifurcation approach may be outlined as follows:

i) A model of the form (2) is assumed, with L(t) a known function of a single variable.

ii) The quasi-static equilibrium $\hat{x}(t)$ is defined implicitly via solutions to (3).

iii) The linearization of (2) evaluated at x(0), $\partial f/\partial x(x(0))$, is assumed stable with all eigenvalues strictly in the left half plane.

iv) The critical value $\hat{t}$ is identified as the smallest positive value of t for which $\partial f/\partial x(\hat{x}(t))$ has eigenvalues crossing the imaginary axis.

To obtain a numerical proximity indicator from this approach, a "distance" to collapse can be defined by the norm size of the load change necessary to cause loss of stability in the linearization, i.e., by $\|L(\hat{t})-L(0)\|$. A Euclidean norm is typically used, but other choices, such as a ℓ-∞ norm, may have more engineering significance. The traditional Q-V curves that define loading limits in a single radial line (sometimes extended to analyze tie line limits in larger systems [22]) are simple single parameter versions of this type of measure. This distance to collapse is expressed in the load parameter space; a similar distance to collapse may be defined in the state space, by calculating $\|\hat{x}(\hat{t})-\hat{x}(0)\|$.

Alternative approaches may relax the assumption of known dependence of L on t, and seek a test to indicate whether any load variation in a set of possible load levels yields a transition to instability for the linearization about $\hat{x}$ (the dependence of $\hat{x}$ on the single parameter t can no longer be assumed). In this case, load levels at individual buses are treated as independent variables, and a set of possible load levels would be defined by examining load vectors within a given weighted norm bound. One then has a corresponding family of possible linearized system matrices, and the goal is to test if there are any unstable matrices in this set. Because the dependence of the linearized matrices on load changes is nonlinear, the set of matrices obtained may be quite complex. However, in [23] it is argued that for some models (similar to those that will be described in Section VI), the set of linearized matrices obtained may be well approximated by a "matrix polytope;" that is, a matrix generalization of a polyhedral set. For the case when the transition to instability must take place with an eigenvalue passing through the origin, this problem is very similar to that of finding singular matrices in an interval matrix family. Recent work on the general interval matrix singularity problem [24] has indicated that the best known algorithms for this problem have computational cost that grows exponentially with the dimension of the matrices and number of free parameters. This suggests that the problem may formally belong to the class of computationally intractable problems that computer science classifies as "NP-hard."

If the problem of deciding whether or not an interval family of matrices has a singular element is NP-hard, a practical algorithm to decide (with complete certainty) whether or not a multiparameter family of power system operating points is guaranteed stable is probably computationally intractable. However, this "curse of dimensionality" is often confronted in power system computations, and sufficiently accurate and reliable approximate algorithms with much lower computational cost can often be found. Hence research into this type of multiparameter bifurcation/small disturbance stability problem will undoubtedly continue.

IV. Sensitivity Approaches

Most of the early, classic works on voltage stability relied on steady state sensitivity analyses, observing that voltage instability could be associated with an operating point at which the ratio of incremental changes in bus voltage to changes in reactive demand approaches infinity [25], [26]. Interestingly, in many models a generalization of this approach yields the same prediction of critical load values as bifurcation analysis, and several more

[1] It is also possible that the growing oscillations predicted by the linearized, small disturbance stability analysis capture only local behavior, and that the actual trajectory enters a limit cycle further away from the old operating point. This aspect of Hopf bifurcation analysis is considered as an explanation for power systems oscillations in [18] and [44].

modern approaches have exploited sensitivity type approaches [27] , [28]. To describe such approaches, consider again (4), which implicitly defines the motion of the quasi-static equilibrium as the load parameters evolve in time. Sensitivity approaches typically treat incremental changes in the load vector ΔL as varying independently about a known operating point x^o and nominal load level L^o. The resulting form of (4) may then be re-written as:

$$0 = f(x^0+\Delta x) + B(x^o+\Delta x)(L^o+\Delta L) \,. \tag{5}$$

Linearizing the relation above about (x^o, L^o) the implicit function theorem predicts the sensitivity of Δx with respect to ΔL as being given by:

$$\Delta x = -\left[\frac{\partial f}{\partial x}(x^o) + \frac{\partial B}{\partial x}(x^o)L^o\right]^{-1} B(x^o)\Delta L \tag{6}$$

When the load parameters are simply P-Q load levels, the matrix B is constant, so the $\partial B/\partial x$ term disappears, leaving

$$\Delta x = -\left[\frac{\partial f}{\partial x}(x^o)\right]^{-1} B\Delta L \tag{7}$$

From (7) it is clear that the state variables, such as voltage magnitudes, can become infinitely sensitive to load levels (and in particular, to reactive load levels) as the system approaches an operating point x^o where $\partial f/\partial x(x^o)$ is singular. But singularity of $\partial f/\partial x(x^o)$ will of course imply that the linearization of (2) about this operating point is on the boundary of stability, with an eigenvalue at the origin. Therefore, while a sensitivity analysis might appear unrelated to the system dynamics, one sees that under standard assumptions on the types of load parameters considered, identifying operating points with infinite sensitivity of state to load changes will simultaneously identify operating points on the boundary of small disturbance stability.

In many cases, such sensitivity analyses can be further simplified to examination only of the powerflow Jacobian, J(x). For example, consider again a simple model that includes only generator swing dynamics and algebraic constraints for power balance at P-Q load buses. The structure of the resulting system of equations will be that of (1), a mixed system of differential equations with algebraic constraints of the form:

$$\dot{\omega} = M^{-1}\{-D\omega - h(\delta, V)\} \tag{8a}$$

$$\dot{\delta}_g = \omega \tag{8b}$$

$$0 = g(\delta, V) \tag{8c}$$

where M represents generator rotational inertias, D generator damping, $h(\delta, V)$ represents active power mismatch at generator buses, and $g(\delta, V)$ represents both active and reactive power balance constraints at load buses. Note that δ_g denotes the subset of phase angles at generator buses, and δ the complete set of bus phase angles. Then (8) may be rewritten as

$$\begin{bmatrix} \dot{\omega} \\ \dot{\delta} \\ 0 \end{bmatrix} = \begin{bmatrix} -M^{-1}D & -M^{-1} & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \omega \\ h(\delta, V) \\ g(\delta, V) \end{bmatrix} \tag{9}$$

It is clear that the linearization of (9) about an operating point will have a zero eigenvalue if the Jacobian of the vector function $[\omega^T, h^T(\delta, V), g^T(\delta, V)]^T$ becomes singular. Defining $x := [\omega^T, \delta^T, V^T]^T$, one has

$$\frac{\partial}{\partial x}\begin{bmatrix} \omega \\ h(\delta, V) \\ g(\delta, V) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & J(x) \end{bmatrix} \tag{10}$$

and it follows that in this type of model, singularity of the powerflow Jacobian at x^o is equivalent to singularity of $\partial f/\partial x(x^o)$, which in turn is equivalent to infinite sensitivity of state variable with respect to certain incremental changes in load parameters, as per (7). This type of relation will be examined in more detail in Section VI.

It is important to note that some recent work employing sensitivity analysis expands these concepts to consider a type of steady state "controllability" of voltage by changes in reactive injections. From an operational standpoint, it is often necessary not only that sensitivity of voltage to changes in injections be bounded, but that these changes have an appropriate sign. For example, it is desirable that an increase in reactive injection from a generator should increase load bus voltages in an associated control area. These requirements are more stringent than simply avoiding singularity of the powerflow Jacobian. Work reported in [27] suggests a variety of computational algorithms for testing this kind of steady state controllability, several of which require the Jacobian to maintain a M-matrix structure.

V. Nonlinear Dynamic Approaches

In describing approaches to voltage collapse that consider nonlinear dynamics, it is important to note that bifurcation analyses do in fact identify transitions in the stability of equilibria for a nonlinear model. However, since bifurcation analysis is based on behavior of the linearization of the dynamics, one expects that this approach will determine primarily local behavior of the trajectories[2]. In examining the behavior in nonlinear dynamic systems with a stable equilibrium point, it is often useful to obtain more global information, characterizing the set of all initial conditions that are attracted back to that equilibrium, the so-called region of attraction. Intuitively, one might associate the "size" (in an appropriate norm or metric) of this region of attraction with the ability of the stable operating point to recover from fast time scale perturbations which tend to disturb the state away from equilibrium. Several of the nonlinear dynamic analyses of voltage collapse which have appeared in the literature have been concerned with identifying regions of attraction.

In [29], a nonlinear model is postulated that focuses exclusively on the dynamics of tap changing transformers, with a decoupled power flow model representing the interchange of power between buses. In this work, time varying parameters of the form $\mathbf{L}(t)$ are not explicitly considered in the dynamic equations, but it is clear that the regions of attraction estimated by [29] will vary with system loading levels. It is also interesting to note that while this model is naturally formulated as differential equations with algebraic constants in the form of (1), assumptions of the decoupled power flow approximation and pure impedance load models makes it possible to solve the algebraic constraints (1b) explicitly for $\mathbf{y}$ in terms of dynamic variables $\mathbf{x}:=\mathbf{z}$. Therefore the authors obtain a set of purely differential equations with the structure (2). This type of model is globally well posed, with no possibility of ill defined trajectories or impasse point behavior. Using only the tap changer dynamics, the authors identify a feasible stable equilibrium in the state space of tap positions, and to construct hyper-rectangles in this state space that approximate the equilibrium's region of attraction. It is interesting to note that the authors of [29] comment that the equilibrium of interest is stable if the associated Jacobian matrix for the dynamic equations, $\partial \mathbf{f}/\partial \mathbf{x}$, is nonsingular. Hence this system model again has the property that transition to instability must take place with an eigenvalue of the linearized dynamics passing through the origin, rather than with complex eigenvalues crossing the imaginary axis. While the question is not explored in [29], it would be interesting to examine whether or not this condition for loss of stability of the equilibrium can be related to singularity of the power flow Jacobian $\mathbf{J}(\mathbf{x})$, as is the case in many other dynamic models.

An alternate nonlinear dynamic approach is an energy based voltage collapse security measure introduced by the authors in [30]. The present paper will review that work by presenting a somewhat different development from that in [30], exploiting an optimal control derivation of energy that makes explicit the role of both slow and fast time scale disturbances, $\mathbf{L}_{slow}(t)$ and $\mathbf{L}_{small}(t)$. This analysis will assume that a suitable dynamic load model or singular perturbation has been employed to obtain a system of the form (2), having no algebraic constraints. An illustration of the singular perturbation approach will be presented in Section VI.

[2]Though again we note that Hopf bifurcation can go somewhat further, and predict nonlinear limit cycle behavior in the vicinity of the old operating point.

The analysis assumes the system is at a known load level and operating point; for simplicity, let these correspond to time t=0, so the operating point is defined by $(\mathbf{L}(0), \mathbf{x}(0))$. Furthermore, assume that this initial operating point has a stable linearization, i.e., $\partial \mathbf{f}/\partial \mathbf{x}(\mathbf{x}(0))$ has all eigenvalues strictly in the left half plane. To isolate changes in load, (2) may be re-written as:

$$\dot{\mathbf{x}} = \left\{ \mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{L}(0) \right\} + \mathbf{B}(\mathbf{x})\Delta\mathbf{L}(t) \tag{11}$$

with $\Delta\mathbf{L}(t)=\mathbf{L}_{small}(t)$ treated as a zero mean, fast time scale load disturbance input to the system. Hence the effect of the load disturbance will not be to change the equilibrium point, but rather to temporarily perturb the state away from the equilibrium $\mathbf{x}(0)$. If $\Delta\mathbf{L}(t)=\mathbf{0}$, the system would remain at the stable equilibrium $\mathbf{x}(0)$. Sufficiently small "impulses" in $\Delta\mathbf{L}(t)$ will result in a state trajectories that asymptotically return to $\mathbf{x}(0)$, but sufficiently large load variations will produce state trajectories that leave the region of attraction about $\mathbf{x}(0)$ and diverge.

As discussed above, the "size" of the region of attraction offers a relative measure of the operating point's ability to recover from such load disturbances. Using an optimal control formulation, one can make this concept much more precise. The impact of load disturbances clearly depends both on their instantaneous magnitude and their duration. Therefore, to measure the size of load disturbances on a time interval [0, T], the following "cost function" is proposed:

$$C_T(\Delta\mathbf{L}(\bullet)) := \int_0^T \|\Delta\mathbf{L}(t)\|_2^2 dt \ . \tag{12}$$

The optimal control problem is then be formulated as follows: for all possible final times T, and all possible piecewise smooth controls $\Delta\mathbf{L}(\bullet)$ that steer the state from $\mathbf{x}(0)$ to a point $\mathbf{x}(T)$ on the boundary of the region of attraction for $\mathbf{x}(0)$, what is the minimum value achieved by $C_T(\Delta\mathbf{L}(\bullet))$? The underlying assumption is that once the state is driven to the boundary of the region of attraction for $\mathbf{x}(0)$, the deterministic dynamics will tend to dominate, and cause the state to diverge. For this phenomena to serve as a plausible description of voltage collapse, one expects that the form deterministic dynamics after the state reaches $\mathbf{x}(T)$ should be such that the voltage magnitudes tend to rapidly decline.

As shown in [31], and subsequently exploited in [30], [32], there exists a class of power system models for which this optimal control problem is solvable in closed form. Indeed, rather than defining only the lowest cost of steering the state from stable equilibrium to points on the boundary of the region of attraction, it is feasible to define a function $\vartheta(x)$ that measures the minimum cost of steering from stable equilibrium $x(0)$ to any other x. Moreover, it is easy to see that the resulting function has the properties of a Lyapunov function with respect to the unperturbed dynamics with $\Delta L(\bullet)\equiv 0$. For example, since the state will not move from $x(0)$ if the control $\Delta L(\bullet)\equiv 0$, there will always be some nonzero cost associated with moving the system off its equilibrium to a neighboring non-equilibrium state, $x\neq x(0)$. Therefore, $\vartheta(x)$ must be locally positive definite about $x(0)$. Also, the function must be nonincreasing along trajectories of the unperturbed system. Suppose there exist an $x(t_1)$ and an $x(t_2)$, $t_1<t_2$, such that a trajectory of the unperturbed system initiated at $x(t_1)$ passes through $x(t_2)$. The cost of steering form $x(0)$ to $x(t_2)$ must be less than or equal than that of steering to $x(t_1)$. To reach $x(t_2)$, one can use the control that steers $x(0)$ to $x(t_1)$, set $\Delta L(t)$ to zero thereafter. The state will eventually reach $x(t_2)$ with no additional control cost. It follows that $\vartheta(x(t_1))\geq\vartheta(x(t_2))$, i.e., the function $\vartheta(x(t))$ is nonincreasing along trajectories $x(t)$ of the unperturbed system.

As shown in [31], the minimum control cost defined by this optimal control approach is equal to a standard Lyapunov function for a system model that include voltage magnitude variation and reactive load models, such as that introduced in [33]. It is also interesting to note that the problem of finding the lowest cost of control to steer the state from stable equilibrium $x(0)$ out of $x(0)$'s region of attraction is closely related to a classic problem in Lyapunov analysis. In particular, the "lowest cost" path from $x(0)$ out of the region of attraction passes through what is termed the "closest unstable equilibrium point," or closest u.e.p. This closest u.e.p may be defined geometrically as the (unstable) equilibrium first encountered by expanding constant contours of $\vartheta(x)$ from $x(0)$. In the optimal control interpretation, a given constant contour represent the set of all points that can be reached with the same cost of control. Using the properties of $\vartheta(x)$ as a Lyapunov function, it is easily shown that these contours must remain closed, bounded, and contained in $x(0)$'s region of attraction until they first encounter another equilibrium point. This equilibrium will prove to be unstable[3], and defines the lowest saddle point on $x(0)$'s region of attraction. Moreover, for the operating conditions and loading patterns associated with voltage collapse, this u.e.p. is typically one that displays very low voltage magnitudes. These low voltage u.e.p.'s are precisely the low voltage power flow solutions that have been exploited in other approaches to detecting the onset of voltage collapse conditions [34], [35], [36]. Trajectories initiated near a low voltage u.e.p., but outside $x(0)$'s region of attraction, will show very rapid decline in voltage magnitude as the state diverges. This will be examined in more detail in the specific system model of Section VI.

As a final comment on modeling in the Lyapunov/optimal control analyses, it is important to note that the closed form solution and true Lyapunov function are only rigorously established for system models with zero transfer conductance and no voltage dependent terms in the active load model. The issue of Lyapunov functions for systems with nonzero conductances has received tremendous attention in the literature, but actually proves a minor point in practical analyses. The controversy in the literature has focused on the question of whether or not it is possible to obtain a *global* Lyapunov function for a system with transfer conductances; that is, a Lyapunov function that remains strictly nonincreasing along *all possible* trajectories, from any initial condition in the state space. This remains a difficult question, with some evidence [37] suggesting that such a global Lyapunov function is impossible to construct for models with transfer conductance. However, in the practical case when one begins with an operating point $x(0)$ whose linearization is strictly stable, it is a trivial matter to construct a *local* Lyapunov function that has the desired properties in a neighborhood of $x(0)$. In [38], a methodology is developed to identify a simple additive correction for the idealized $\vartheta(x)$ that results from the exact, path independent case. The result yields a *local* Lyapunov function that satisfies the required conditions in a fairly large neighborhood of the stable equilibrium $x(0)$. While not true *global* Lyapunov functions, these energy[4] functions display behavior that is approximately nonincreasing along trajectories initiated in $x(0)$'s region of attraction, and prove quite reliable in estimating this region of attraction [32]. Alternative approaches to deriving practically useful energy functions for for systems models lacking a path independent integral have been in use for many years [39].

To relate this type of energy analysis back to steady state and bifurcation analysis, the role of fast versus slow time scale load variations becomes important. Bifurcation analyses examine load variations that occur on time scales slower than the underlying system dynamics, and identify how far a given load increase can be "pushed" until the operating point looses stability. The energy analysis begins with a stable operating point obtained when the slow load variation is frozen at a value below the critical threshold. For that operating point $x(0)$, the energy function evaluated at the closest u.e.p identifies the smallest fast time scale load disturbance that can push the state out of $x(0)$'s region of attraction, causing the state to then diverge. This energy margin essentially measures the size of the region of attraction, but does so using a metric that is closely related to the dynamics of the system: the solution to the

[3]This statement is true for systems of the form assumed in [D&B], but again, care is necessary in analyzing mixed systems of differential equations with algebraic constraints. The equilibrium first encountered by expanding contours of the same function $\vartheta(x)$ can prove to be stable in mixed differential/algebraic models.

[4]The convention here will be to use the term energy function when the resulting expression can not be proven to be nonincreasing along all possible trajectories

optimal control problem. The limit of the energy margin agrees with the limits predicted by a bifurcation analysis in that as the slow load variation pushes the operating point towards a value where it looses small disturbance stability, the energy margin goes to zero. The bifurcation analysis for this type of model predicts that the operating point x(0) must merge with an unstable saddle point (a type-one u.e.p.) at the critical loading level. This unstable equilibrium is precisely the closest u.e.p. in energy analysis; when the closest u.e.p. and operating point merge, it is obvious that the energy margin between the two must shrink to zero.

As a final note on the relation of slow and fast time scale phenomena in examining voltage collapse, the recent work of [43] is noteworthy. In that work, the authors do not separate time scales of exogenous load variations, as has been done here, but rather separate time scales in the dynamic equations themselves. This leads to a partition of the state variables (possibly in a new coordinate system) into slow and fast sets, and an approximate decoupling of the slow and fast dynamic subsystems. Then separate stability analyses can be performed for the decoupled slow and fast subsystems, allowing separate examination of instability associated with slower dynamics such as tap changing transformers, and faster dynamics of generator excitation dynamics coupled to load characteristics. This type of integrated analysis may ultimately allow coupling between the type of model described here, and the tap changing transformer dynamic analysis of [29].

VI. Sample System Model and Comparison of Proximity Measures

To illustrate the relationships between analysis methods in more detail, it is not necessary to examine specific numerical examples, but rather the class of models must be considered. To this end, this section will develop of model of the structure of (1) that has many of the features considered in voltage stability analyses, and will allow explicit comparisons between methods of the types described in preceding sections. We will then discuss a methodology for modifying this mixed system of differential equations with algebraic constraints to obtain a system of purely differential equations. Also, the nature of ill-posed trajectories in the mixed differential/algebraic systems will be reviewed, and the relation of this problem to load modelling discussed. Finally, the relation between sensitivity, bifurcation and energy analyses will be illustrated in detail for the differential equation model.

It is also important to note the types of analyses that are not captured in such models. Only an approximate representation of reactive output saturation of generation is represented; no detailed exciter dynamics are included. Therefore, approaches to voltage stability improvement through improved nonlinear control schemes for excitation, such as that proposed in [40], is beyond the scope of this model. Similarly, loss of small disturbance stability via complex eigenvalues associated with exciter dynamics migrating across the imaginary axis, as discussed in [18] and [19], is also not considered.

To illustrate the structure discussed in the preceding sections for a specific model, consider the following possible representation. The differential equations represented will arise from simple swing dynamics for generators, and from active loads have a component that is proportional to bus frequency, as originally developed for Lyapunov analysis in [41]. Linear frequency dependent loads offer a simple approximation of behavior of various types of motor loads. Reactive loads are allowed to be an arbitrary polynomial or exponential function of bus voltage magnitude; denote this function $Q_i(V_i)$ The expression for reactive power absorbed by the network can be found in any standard text treating power flow analysis, and can be written as a function of the vector $\alpha \in \mathbf{R}^{n-1}$ of phase angles (relative to a reference bus), and the vector $V \in \mathbf{R}^n$ of bus voltage magnitudes; denote this expression at bus i as $\tilde{g}_i(\alpha, V)$. The resulting reactive power balance equation becomes:

$$0 = Q_i(V_i) - \tilde{g}_i(\alpha, V) \tag{13}$$

In some types of analyses, it proves convenient to first normalize (13), dividing the equality by V_i^{-1} . The resulting Jacobian of the associated power balance equations becomes nearly symmetric, which can be advantageous both for certain numerical algorithms and for energy type analyses. With this normalization, the reactive power balance equations then have the structure

$$0 = V_i^{-1}\left\{ Q_i(V_i) - \tilde{g}_i(\alpha, V) \right\}, \tag{14}$$

which then make up the algebraic constraints shown by (1b) in the general model. The L(t) terms would appear as variations in the terms of $Q_i(V_i)$; these can be made explicit when necessary.

Many authors have commented on the importance of reactive limits on generators in the voltage collapse phenomenon. The structure of equation in (14) may be easily adapted to model this effect. Assume that the generator excitation system at bus i is normally in a voltage control mode, with voltage setpoint V_i^0. A simple representation of the exciter's behavior would increase the generators MVar output when bus voltage drops below V_i^0, and decrease its output when bus voltage rises above V_i^0. However, the reactive output has both upper and lower limits, so if the reactive power absorbed by the network exceeds the maximum output of the generator, the bus voltage will go out of its "control band" about V_i^0. This effect can be approximately represented by choosing $Q_i(V_i)$ for the generator as shown in Figure 1 below.

With this assumed form of reactive power balance at a generator buses, the effect of reactive limits can be included in an algebraic constraint of the same form as (14). It should be clear that the characteristic given in Figure 1 is really just a "smoothing" of the familiar switching function between PV bus behavior and PQ bus behavior that is typically used to represent reactive limits in power flow analysis. In the limit of a step change between Q_{imax} and Q_{min}, one recovers the PV/PQ switching behavior. However, for small disturbance stability analyses that require differentiation of the characteristic, the smoothed representation in Figure 1 appears a more tractable and equally plausible approximation. With the smoothed approximation, the generator terminal bus voltage magnitude remains a variable in all ranges of operation, so the dimension of the power flow Jacobian is not changed as the generator reaches its reactive limit.

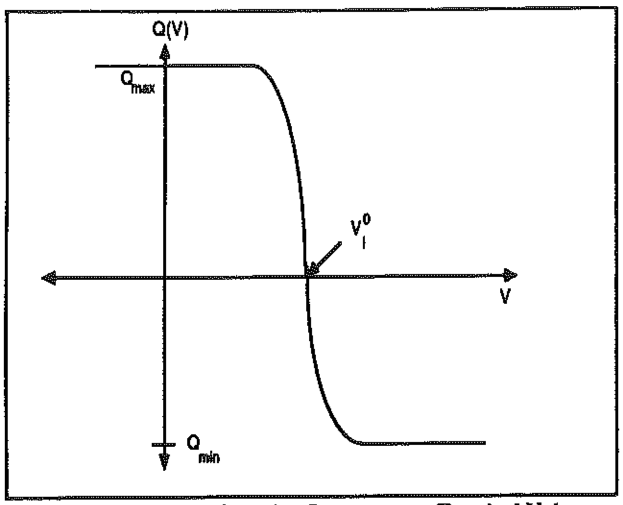

Figure 1: Generator Reactive Output versus Terminal Voltage

With the individual component models described above, the overall system model can be assembled, with variables given by:

$\omega \in \mathbb{R}^m$, generator deviation from synchronous frequency,
$\alpha \in \mathbb{R}^{n-1}$, bus voltage phase angles relative to reference bus,
$V \in \mathbb{R}^n$, bus voltage magnitudes.

The differential/algebraic equations for this power system model are then written as:

$$\dot{\omega} = -M^{-1}T_1^T f(\alpha, V) \tag{15a}$$

$$\dot{\alpha} = T_1\omega + T_2 D_l^{-1} T_2^T f(\alpha, V) \tag{15b}$$

$$0 = g(\alpha, V) \tag{15c}$$

where:

$$f(\alpha, V) := \tilde{f}(\alpha, V) + P^0$$

$\tilde{f}_i(\alpha, V)$:= active power absorbed by network at bus i

P_i^0 := nominal active power demanded by load at bus i

$$g(\alpha, V) := \{\text{diag}[V]\}^{-1}\{\tilde{g}(\alpha, V) + Q_D(V)\}$$

$\tilde{g}_i(\alpha, V)$:= reactive power absorbed by network at bus i

$Q_{D,i}(V_i)$:= reactive power demanded by load at bus i (or -generated Q)

and M, D_l, D_V are constant diagonal matrices describing system parameters, and T_1 and T_2 are constant matrices describing network topology.

As mentioned above, differential equations with algebraic constraints may not produce well defined trajectories from all initial conditions. In particular, they may produce trajectories that exist over some time interval $[0, \hat{t}]$, but become ill-defined for $t > \hat{t}$. A trivial example of this occurs in a two dimensional (non power system) example of $dx/dt = x+y$; $0=x^2+y^2-1$. Initiating this system at (x=0, y=1), it is easily verified that the trajectories evolve on the circle defined by the algebraic constraint, until the point (x=1, y=0) is reached. At this impasse point, x can no longer increase while still remaining on the circle, and the trajectory can not be continued. In a numerical integration that alternates between updating dynamic variables and solving algebraic constraints, this problem would show up as a new value of x being generated for which no y solution exists. Reference [7] verifies that this type of impasse behavior is easily constructed for simple power system examples when the reactive power balance is modelled as an algebraic constraint, and the load has some non-zero constant Q component (though it need not be purely a constant Q load).

The impasse point problem disappears if *all* loads are modelled as purely constant impedances, since in this case it is always possible to explicitly solve for the algebraic variables in terms of dynamic variables. This problem is tacitly acknowledged in numerical simulation approaches; for example, the review paper [2] comments that for load models containing constant power terms, solving the algebraic constraints may become numerically ill conditioned (a sign of the onset of impasse point behavior) when voltages dip. In [2] it is suggested that the simulation instantaneously "switch" to a constant impedance model. While such an approach may be physically motivated by the phenomena of stalling motors, it is clear that a more precise description of such transitions would be needed for reasonable voltage stability analysis. Moreover, it is clear that analytically studying the behavior of large systems of differential equations with algebraic constraints can be even more challenging than numerically solving for specific trajectories.

To eliminate the problem of algebraic constraints while maintaining consistent behavior in the vicinity of the operating point, the algebraic constraints may be singularly perturbed to produce a differential equation for the algebraic variables. In the case of the bus voltage magnitudes and reactive power balance equations, this approach produces equations of the form:

$$\dot{V} = \frac{1}{\varepsilon} V_i^{1} \left\{ Q_i(V_i) - g_i(\alpha, V) \right\} . \qquad (16)$$

From an engineering standpoint, (16) may be interpreted as follows. The demand term is taken as the "independent input," and the voltages respond to this input to maintain reactive power balance. The right hand side of (16) is the difference between reactive power delivered by the network and reactive power absorbed by the load (or vis versa for the generator case). When the load instantaneously demands more reactive power than is being delivered, (16) predicts that the bus voltage drops until power balance is re-established. When excess reactive power is delivered by the network, bus voltage magnitude will increase. The rate of this change varies with ε; for ε sufficiently small, it is essentially instantaneous and behavior is nearly identical to the original static model. Clearly, the equilibria of (16) are identical to the solutions of (14). To obtain consistent behavior between a model based on (14) and that of a model based on (16), standard results in singular perturbations (described in [7]) require that the linearization of (16) be stable in a neighborhood about the solution of interest for (14) (i.e., the solution of the full powerflow). Satisfaction of this condition depends in part on the exact form of the voltage dependence of the load. This analysis to confirm stability for a particular load model is left to the reader; here we will simply state without proof that experience with a wide range of load models shows the linearization of (16) about reasonable (per unit voltage magnitude above 0.7) operating points is typically stable. However, behavior of (16) differs from the model with algebraic constraints if one examines stability in the vicinity of "low voltage' power flow solutions. The algebraic constraint combined with standard small disturbance stability models would predict these equilibria to be stable; the differential model (16) predicts such points to be unstable. Unfortunately, because such low voltage solutions are usually not operable for other reasons, it is difficult to confirm which prediction best matches observed physical behavior. The last point concerning this model relates to the choice of ε. Clearly, this time constant should be small (probably less than 0.01 second) to match observed voltage behavior in power systems; fortunately, the energy function does not depend on the choice of ε.

For differential equations to define generator terminal voltage under the reactive generation limit model described above, the form of (16) is kept, with Q(V) reinterpreted as the reactive power supplied by the generator. For this interpretation of (16), ε should be chosen as the time constant associated with the response of the generators reactive output with respect to terminal voltage regulation errors. Again, because ε does not enter the energy function explicitly, the exact choice of this parameter is not critical.

With the singular perturbation assumptions above, the modified state model is composed purely of differential equations, and takes the form

$$\dot{\omega} = -M^{-1} T_1^T f(\alpha, V) \qquad (17a)$$

$$\dot{\alpha} = T_1 \omega + T_2 D_1^{-1} T_2^T f(\alpha, V) \qquad (17b)$$

$$\dot{V} = -D_V g(\alpha, V) \qquad (17c)$$

with the new parameter matrix D_V simply being a constant diagonal matrix composed of the ε_i terms.

For a network with zero transfer conductances, and for $f(\alpha, V)$ defined with active loads having no voltage dependent component, the composite vector function $[(M\omega)^T, f^T(\alpha, V), g^T(\alpha, V)]^T$ is exact. Hence the value of a path integral of this function from $(0, \alpha^0, V^0)$ to (ω, α, V) will depend only on the endpoints. Therefore one may define

$$\vartheta(x^0, x) := \int_{(0,\alpha^0,V^0)}^{(\omega,\alpha,V)} \left[(M\lambda)^T, f^T(\xi,\mu), g^T(\xi,\mu)\right] \left[d\lambda^T, d\xi^T, d\mu^T\right]^T \qquad (18)$$

which under the stated assumption on the network and active power loads may be re-expressed as

$$\frac{1}{2}\omega^T M \omega - \frac{1}{2}\sum_{i=0}^{n}\sum_{k=0}^{n} B_{ik} V_i V_k \cos(\alpha_i - \alpha_k)$$
$$+ \frac{1}{2}\sum_{i=0}^{n}\sum_{k=0}^{n} B_{ik} V_i^0 V_k^0 \cos(\alpha_i^0 - \alpha_k^0)$$
$$+ \sum_{k=0}^{n} \int_{V_k^0}^{V_k} \frac{Q_{D,k}(\mu)}{\mu} d\mu - \left[P^0\right]^T (\alpha - \alpha^0) \qquad (19)$$

In the case where the network is not lossless, and/or the active load does have a voltage dependent component, the integral given in (18) is not path independent. In this case, a simple linear correction term added to $\vartheta(x^0, x)$ makes the function satisfy the criteria formal criteria for a Lyapunov function in the vicinity of the stable equilibrium, as described in [32]. The discussion to follow will assume the exact case for illustrative purposes. To illustrate the close connection between the energy function and the dynamic model, we define

$$A = \begin{bmatrix} 0 & -M^{-1}T_1^T & 0 \\ T_1M^{-1} & -T_2D_1^{-1}T_2^T & 0 \\ 0 & 0 & -D_V^{-1} \end{bmatrix} \quad (20)$$

With these definitions, the power system dynamics of (17) may be rewritten

$$\dot{x} = A\,\nabla_x\vartheta(x^0, x) \quad (21)$$

By construction, A is nonsingular and negative semi-definite. In particular, A is the sum of two matrices: a diagonal matrix of non-positive terms, and a skew symmetric matrix. Roughly speaking, the right hand side of (21) can be thought of as a sum of a gradient like term (diagonal matrix times $\nabla\vartheta$) and a Hamiltonian like term (skew symmetric matrix times $\nabla\vartheta$). Given that A is nonsingular, equilibria of (21) occur only at points where $\nabla_x\vartheta(x^0, x)=0$. Furthermore, if the Hessian of ϑ about an equilibrium is positive definite, using LaSalle's theorem with ϑ as a candidate Lyapunov function it may be shown that that equilibrium must be asymptotically stable.

The relation between the gradient of energy and the equilibria of the dynamic equations provides a first link between energy analysis and steady state criteria. Suppose that a slow time scale variation in load parameters is considered, of the general form suggested in (4), but with the load variations appearing as additive P-Q terms. Then (17) is modified to become

$$\dot{\omega} = -M^{-1}T_1^T f(\alpha, V) \quad (22a)$$
$$\dot{\alpha} = T_1\omega + T_2D_1^{-1}T_2^T(f(\alpha, V) + P(t)) \quad (22b)$$
$$\dot{V} = -D_V g(\alpha, V) + Q(t) \quad (22c)$$

The equation for the quasi-static evolution of the equilibrium, $\hat{x}(t)$, can then be written as:

$$0 = A\left\{ \nabla\vartheta(x^0, \hat{x}(t)) + BL(t) \right\} \quad (23)$$

where

$$B := \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix};\; L(t) := \begin{bmatrix} 0 \\ P(t) \\ Q(t) \end{bmatrix}.$$

Again noting that the matrix A is nonsingular, the quasi-static evolution of the equilibrium is entirely determined by

$$0 = \nabla\vartheta(x^0, \hat{x}(t)) + BL(t) \quad (24)$$

Moreover, when partitioned into three blocks consistent with (ω, α, V), the second and third blocks of $\nabla\vartheta$ are simply the power balance equations. Hence one has the unsurprising result that the only parts of the equilibrium that evolve with changing load are the bus phase angles and voltage magnitudes, as predicted by the power flow.

The close association of the power flow Jacobian to the linearized dynamics can also be easily illustrated in this model. The linearization of the dynamics about an equilibrium $\hat{x}(t)$ is determined by the Hessian of ϑ, and is given by

$$\dot{x} = A\frac{\partial^2}{\partial x^2}\vartheta(\hat{x}(t), \hat{x}(t)) \;. \quad (25)$$

But from the construction of the energy function ϑ, it follows (for the idealized exact case) that

$$\frac{\partial^2}{\partial x^2}\vartheta(\hat{x}(t), \hat{x}(t)) = \begin{bmatrix} M & 0 \\ 0 & J(\hat{x}(t)) \end{bmatrix} . \quad (26)$$

Recalling that M is a diagonal matrix of generator inertia constants, the Hessian of the energy is positive definite if and only if the Jacobian of the power flow is positive definite. Moreover, from (25) and (26) it is clear that a singular power flow Jacobian must produce a zero eigenvalue of the linearized dynamics. One concludes that the linearization is stable if and only if the power flow Jacobian is positive definite, and is on the boundary of small disturbance stability when the power flow Jacobian becomes singular. Hence, a quasi-static bifurcation analysis to determine when the equilibrium $\hat{x}(t)$ looses small disturbance stability is reduced to testing when $J(\hat{x}(t))$ looses strict positive definiteness and becomes singular.

Specializing the relation (7) to this model also confirms that the quasi-static solution for the operating point becomes infinitely sensitive to changes in P-Q load levels as $J(\hat{x}(t))$ approaches singularity. Hence, for the model examined here, tests for infinite sensitivity of voltages to loads predict the same critical loading value as a bifurcation analysis to detect when the operating point looses small disturbance stability. The energy/optimal control based security measure also goes to zero as the operating point approaches the condition of singular $J(\hat{x}(t))$.

The above discussion confirms that the three basic methods all predict the same critical loading level for the quasi-static load variation, $L_{slow}(t)$. However, when the nominal load has not yet reached a level where small disturbance stability is lost, the manner by which different methods measure the ***proximity*** to collapse can vary widely. As noted earlier, some methods use the smallest singular value of the power flow Jacobian. If applied to models that allow discontinuous switching of generators from PV to PQ representations, the Jacobian will display a change of dimension when the switching occurs, and the smallest singular value will then be discontinuous with respect to changes in load, as was illustrated for an IEEE 30 bus test system example in [42]. While the continuous model for reactive saturation suggested in Figure 1 would eliminate the discontinuity, eigenvalues and singular values remain highly nonlinear functions of the elements of $J(\hat{x}(t))$, so this proximity measure would still be expected to display very nonlinear behavior. Sensitivity based measure that attempt to track the magnitude of certain dV/dQ terms as $J(\hat{x}(t))$ evolves will also tend to display very nonlinear behavior, and may vary widely from bus to bus. The plots of such measures provided in [42] are very informative in this regard. As an alternative, approaches that use the norm margin of load variation until collapse, of the form $\|L(\hat{t})-L(0)\|$, will tend to be smoother, more nearly linear function of t. However, these measures may be very sensitive to the assumed functional dependence of L on t. In physical terms, this is the assumed pattern of load increase. It is clear that a pattern of load increase that puts greater load on buses with weak reactive support will push the system to collapse more rapidly, and lead to a smaller margin. This is a desirable characteristic for the proximity measure, *provided the predicted pattern of load increase is sufficiently accurate*. In an operating environment where the load projections may have significant margins of error, and operators wish to determine the risk of voltage problems on a time horizon of several hours, this could prove problematic. The obvious alternative is to look for a worst case pattern of load increase (perhaps with some constraints to keep the studied load patterns realistic), and judge proximity to collapse by the minimum, worst case distance. However, from a computational standpoint, this reduces again to the potentially intractable problem of testing for singularity in a multiparameter family of Jacobian matrices.

The energy/optimal control approach offers a more computationally tractable "worst case" proximity measure, but it is able to do so because of a different assumption on the nature of the load variation. The quasi-static measures seek to identify the worst case variation in $L_{slow}(t)$ such that the solution to (4) yields an operating point on the boundary of small disturbance stability, or equivalently, with infinite sensitivity of state to further load variations. The energy measure assumes that the current operating point defines a *fixed* equilibrium for the dynamic model, and the load variations of interest are on a fast time scale that excites the dynamics and moves the state temporarily away from equilibrium. In this formulation, the "worst case" fast time scale load variation $\Delta L(t)$ is in fact the solution to the optimal control problem that drives the system state out of the equilibrium point's region of attraction. As described in Section V, the energy based proximity measure is then an integral of the norm squared of this $\Delta L(t)$ load variation with respect to time. It is important to stress that once the operating point and load level under study are set, *no assumed pattern of quasi-static load evolution ($L_{slow}(t)$) is necessary for the energy calculation*. This is perhaps the most critical distinction between the quasi-static measures and the energy based measure, and again stresses the need for careful understanding of the assumed time scales when comparing various voltage collapse proximity measures.

VII. Conclusions

This paper has compared the a range of voltage collapse analysis techniques that include both quasi-static bifurcation and sensitivity measures, and nonlinear dynamic measures. For a reasonable power system model incorporating many of the characteristics relevant to voltage collapse, and an assumption of slow time scale, quasi-static evolution of loads, it is shown that all such measures predict the same ultimate loading limit. However, if the load variations of interest include both a slow moving "average" term, and a faster time scale zero mean "disturbance" term, it is argued that a measure of proximity to collapse offered by nonlinear analyses will contain additional information. It is hoped that the close connections between various analysis techniques can be exploited, so that the various security measures can be used in complementary fashion in actual applications.

VIII. Acknowledgements

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