ECEN 615
Methods of Electric Power Systems Analysis

Lecture 21: Security Constrained Optimal Power Flow (SCOPF), Convex Relaxation, Power Markets

Prof. Tom Overbye
Dept. of Electrical and Computer Engineering
Texas A&M University
overbye@tamu.edu

Announcements

- Homework 5 is due on November 13.
- The second exam is on Thursday December 4 during class (Sanjana will make arrangements for the distance education students).
  - Closed book, closed notes. Calculators and two 8.5 by 11 inch note sheets are allowed
- The design project should be turned in by Monday Dec 8 (i.e., the last day of classes for the semester).
- A good, free book on Convex Optimization by Boyd and Vandenberghe is available at web.stanford.edu/~boyd/cvxbook/bv_cvxbook.pdf

DC Optimal Power Flow

- For simplicity and computational efficiency in the OPF, the dc power flow is often used to replace ac power flow, leading to the dc OPF model.
- The model is significantly simplified because voltage, reactive power, and power losses are all ignored.

$$
\begin{array}{lll}
\text{Obj.} & \min \sum_{i \in \mathcal{N}_G} c_i^G(P_i^G) & \\
\text{s.t.} & \underline{P}_i^G \le P_i^G \le \bar{P}_i^G, & i \in \mathcal{N}_G \\
& P_i^G - P_i^D = \sum_{j: ij \in \mathcal{E}} p_{ij}, & i \in \mathcal{N} \\
& p_{ij} = \frac{1}{x_{ij}}(\theta_i - \theta_j), & ij \in \mathcal{E} \\
& \underline{\theta}_i \le \theta_i \le \bar{\theta}_i, & i \in \mathcal{N} \\
& \underline{p}_{ij} \le p_{ij} \le \bar{p}_{ij}, & ij \in \mathcal{E}
\end{array}
$$

- One of my most cited papers ([a]) looks at a comparison between the ac and dc OPFs, with the conclusion being “it depends.”
- You’ll be using the dc OPF in your design project.

[a] T.J. Overbye, X. Cheng, Y. San, “A Comparison of the AC and DC Power Flow Models for LMP Calculations,” Proc. 37th Hawaii International Conf. on System Sciences, 2004

Multi-Time Period DCOPF

- The previous OPF models are single-period, only optimizing one time snapshot. Both ac OPF and dc OPF can directly extend to multi-period settings, to capture the time-varying system conditions.
- For example, a multi-period dc OPF is formulated considering the time horizon t ={1,2, …, T}.
- Multi-period OPF involves time-coupled constraints, such as the generator ramping constraints.

$$
\begin{array}{lll}
\text{Obj.} & \min \displaystyle\sum_{t=1}^{T} \sum_{i \in \mathcal{N}_G} c_{i,t}^{G}(P_{i,t}^{G}) & \\
\text{s.t.} & \underline{P}_i^G \leq P_{i,t}^G \leq \bar{P}_i^G, & i \in \mathcal{N}_G, t \in \mathcal{T} \\
 & \underline{\Delta}_i^G \leq P_{i,t}^G - P_{i,t-1}^G \leq \bar{\Delta}_i^G, & i \in \mathcal{N}_G, t \in \mathcal{T} \\
 & P_{i,t}^G - P_{i,t}^D = \displaystyle\sum_{j:ij \in \mathcal{E}} p_{ij,t}, & i \in \mathcal{N}, t \in \mathcal{T} \\
 & p_{ij,t} = \dfrac{1}{x_{ij}}(\theta_{i,t} - \theta_{j,t}), & ij \in \mathcal{E}, t \in \mathcal{T} \\
 & \underline{\theta}_i \leq \theta_{i,t} \leq \bar{\theta}_i, & i \in \mathcal{N}, t \in \mathcal{T} \\
 & \underline{p}_{ij} \leq p_{ij,t} \leq \bar{p}_{ij}, & ij \in \mathcal{E}, t \in \mathcal{T}
\end{array}
$$

Renewable Generation and Energy Storage

- For Renewable Generation, such as solar and wind, two operational settings are widely used:
  - Non-dispatchable: treated as negative load
  - Dispatchable $0 \leq P_{i,t}^{R} \leq \bar{P}_{i,t}^{R}$
- For Energy Storage, its model is formulated at

Here the P^{ds} values are the amount of power being supplied, and the P^{ch} the amount for charging; E is the energy stored in the device.

$$P_{i,t}^{ES} = P_{i,t}^{ds} - P_{i,t}^{ch}$$

$$0 \leq P_{i,t}^{ds} \leq \bar{P}_{i}^{ds}, \; 0 \leq P_{i,t}^{ch} \leq \bar{P}_{i}^{ch}$$

$$P_{i,t}^{ds} \cdot P_{i,t}^{ch} = 0$$

$$E_{i,t} = \kappa_i E_{i,t-1} - \Delta t \big(\tfrac{1}{\eta_i^{ds}} P_{i,t}^{ds} - \eta_i^{ch} P_{i,t}^{ch} \big)$$

$$\underline{E}_i \leq E_{i,t} \leq \bar{E}_i$$

Security Constrained OPF

- Security constrained optimal power flow (SCOPF) is similar to OPF except it also includes contingency constraints.
  - Again the goal is to minimize some objective function, usually the current system cost, subject to a variety of equality and inequality constraints
  - This adds significantly more computation, but is required to simulate how the system is actually operated (with N-1 reliability)
- A common solution is to alternate between solving a power flow and contingency analysis, and an LP.

Security Constrained OPF, cont.

- With the inclusion of contingencies, there needs to be a distinction between what control actions must be done pre-contingent, and which ones can be done post-contingent.
  - The advantage of post-contingent control actions is they would only need to be done in the unlikely event the contingency actually occurs
- Pre-contingent control actions are usually done for line overloads, while post-contingent control actions are done for most reactive power control and generator outage re-dispatch.

OPF and SCOPF History

- A nice OPF history from Dec 2012 is provided by the below link:

https://www.ferc.gov/sites/default/files/2020-05/acopf-1-history-formulation-testing.pdf (by M Cain, R. O’Neill, A. Castillo)

- Prior to digital computers economic dispatch was solved by hand and the power flow with network analyzers.
- Digital power flow developed in late 50’s to early 60’s.
- First OPF formulations in the 1960’s.
  - J. Carpienterm, “Contribution e l’étude do Dispatching Economique,” Bulletin Society Francaise Electriciens, 1962
  - H.W. Dommel, W.F. Tinney, “Optimal power flow solutions,” *IEEE Trans. Power App. and Systems*, Oct. 1968
    - “Only a small extension of the power flow program is required”

OPF and SCOPF History, cont.

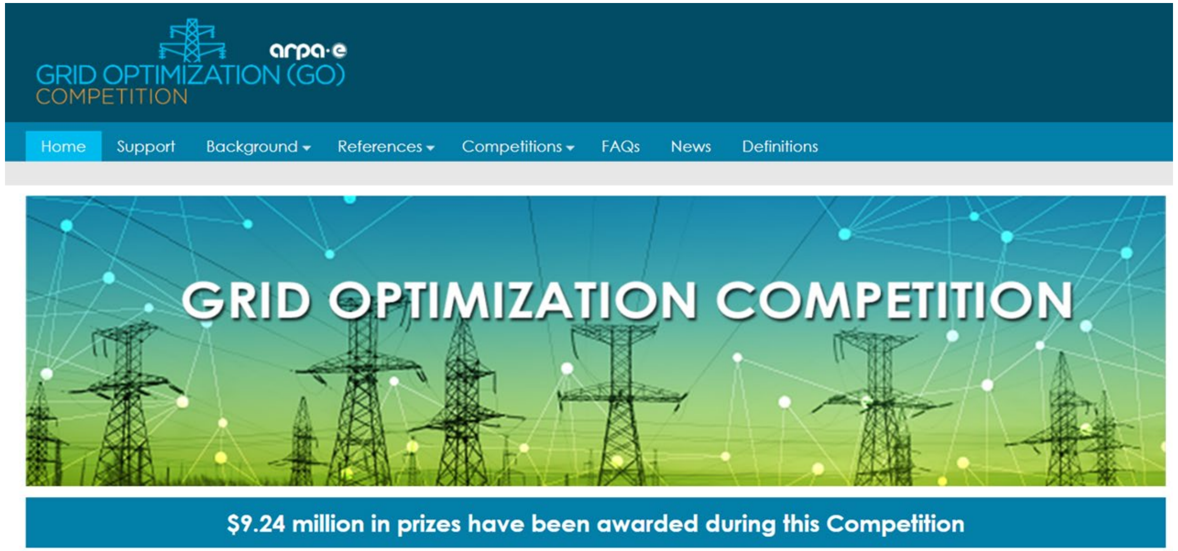
- An LP approach was presented by Stott and Hobson in 1978
  - B. Stott, E. Hobson, “Power System Security Control Calculations using Linear Programming,” (Parts 1 and 2) *IEEE Trans. Power App and Syst.*, Sept/Oct 1978
- Optimal Power Flow By Newton’s Method
  - D.I. Sun, B. Ashley, B. Brewer, B.A. Hughes, and W.F. Tinney, "Optimal Power Flow by Newton Approach", *IEEE Trans. Power App and Syst.*, October 1984
- Follow-up LP OPF paper in 1990
  - O. Alsac, J. Bright, M. Prais, B. Stott, “Further Developments in LP-based Optimal Power Flow,” *IEEE Trans. Power Systems*, August 1990
- Critique of OPF Algorithms
  - W.F. Tinney, J.M. Bright, K.D. Demaree, B.A. Hughes, “Some Deficiencies in Optimal Power Flow,” *IEEE Trans. Power Systems*, May 1988

OPF and SCOPF History, cont.

- The SCOPF is presented by Monticelli, Pereira and S. Granville in
  - A. Monticelli, M. V. F. Pereira and S. Granville, "Security-Constrained Optimal Power Flow with Post-Contingency Corrective Rescheduling," in IEEE Transactions on Power Systems, vol. 2, no. 1, pp. 175-180, Feb. 1987.
- A nice coverage of a full SCOPF implementation is given in
  - T. J. Bertram, K. D. Demaree and L. C. Dangelmaier, "An integrated package for real-time security enhancement," in *IEEE Transactions on Power Systems*, vol. 5, no. 2, pp. 592-600, May 1990
- Tens of thousands of other papers on OPF and SCOPF!

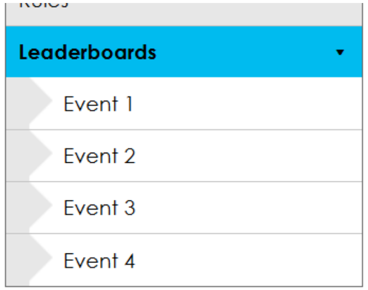
OPF and SCOPF Current Status

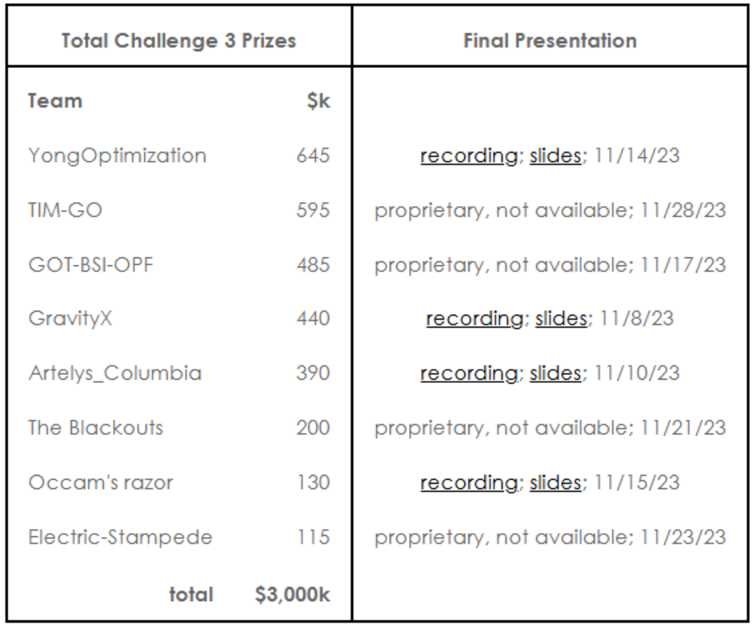
- OPF (really SCOPF) is currently an area of active research, with DOE ARPA-E just recently completing the GO (Grid Optimization) Competition
  - Details on the initial competition are given at arpa-e.energy.gov/programs-and-initiatives/view-all-programs/go-competition
  - There were a total of three competitions, with Texas A&M being a data set provider for all of them.



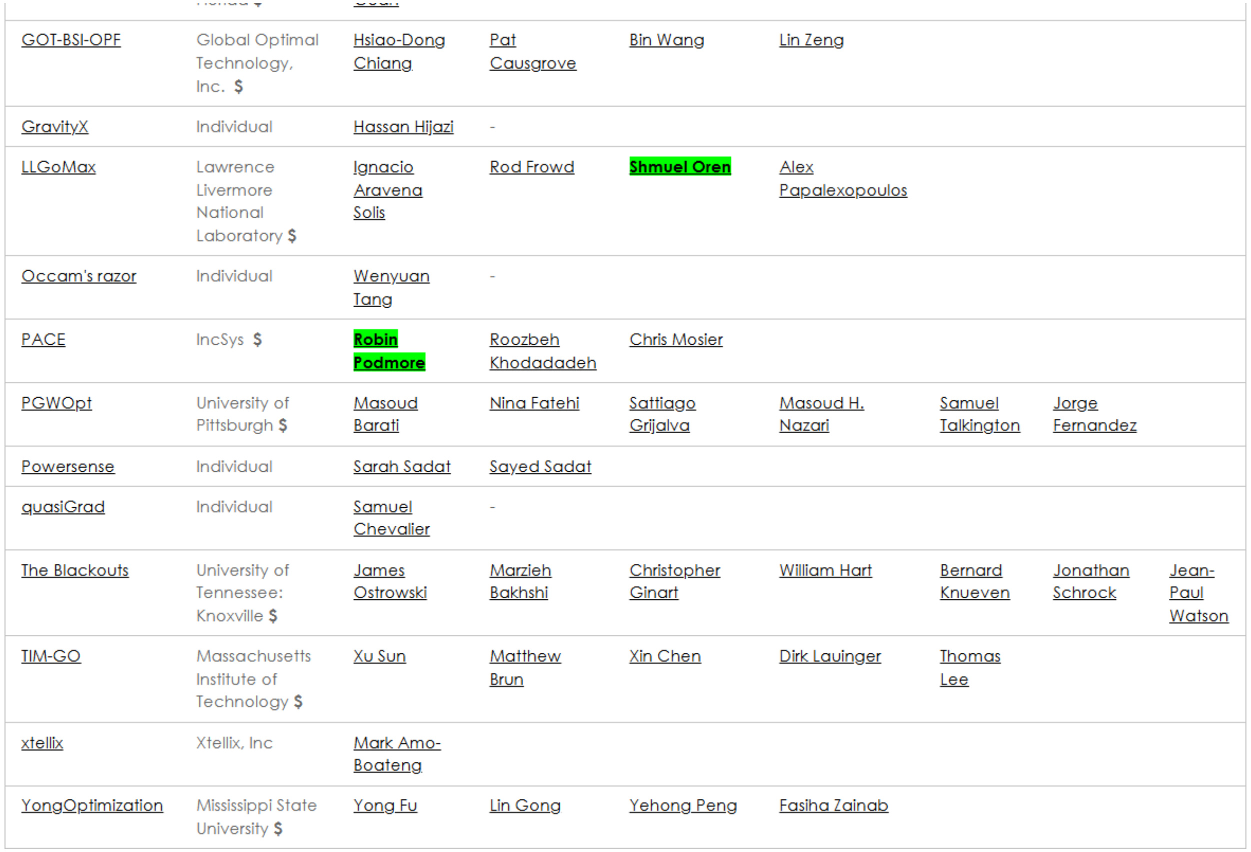
$9.24 million in prizes have been awarded during this Competition

Prof. Xin’s Team took 2nd Place in Challenge 3 in 2023!





Total Challenge 3 Prizes		Final Presentation
Team	**$k**	
YongOptimization	645	recording; slides; 11/14/23
TIM-GO	595	proprietary, not available; 11/28/23
GOT-BSI-OPF	485	proprietary, not available; 11/17/23
GravityX	440	recording; slides; 11/8/23
Artelys_Columbia	390	recording; slides; 11/10/23
The Blackouts	200	proprietary, not available; 11/21/23
Occam's razor	130	recording; slides; 11/15/23
Electric-Stampede	115	proprietary, not available; 11/23/23
total	**$3,000k**	



GOT-BSI-OPF	Global Optimal Technology, Inc. $	Hsiao-Dong Chiang	Pat Causgrove	Bin Wang	Lin Zeng			
GravityX	Individual	Hassan Hijazi	-					
LLGoMax	Lawrence Livermore National Laboratory $	Ignacio Aravena Solis	Rod Frowd	**Shmuel Oren**	Alex Papalexopoulos			
Occam's razor	Individual	Wenyuan Tang	-					
PACE	IncSys $	**Robin Podmore**	Roozbeh Khodadadeh	Chris Mosier				
PGWOpt	University of Pittsburgh $	Masoud Barati	Nina Fatehi	Sattiago Grijalva	Masoud H. Nazari	Samuel Talkington	Jorge Fernandez	
Powersense	Individual	Sarah Sadat	Sayed Sadat					
quasiGrad	Individual	Samuel Chevalier	-					
The Blackouts	University of Tennessee: Knoxville $	James Ostrowski	Marzieh Bakhshi	Christopher Ginart	William Hart	Bernard Knueven	Jonathan Schrock	Jean-Paul Watson
TIM-GO	Massachusetts Institute of Technology $	Xu Sun	Matthew Brun	Xin Chen	Dirk Lauinger	Thomas Lee		
xtellix	Xtellix, Inc	Mark Amo-Boateng						
YongOptimization	Mississippi State University $	Yong Fu	Lin Gong	Yehong Peng	Fasiha Zainab			

Comprehensive Grid Optimization Model

- The GO Competition provides a very comprehensive problem formulation for security-constrained ACOPF (SC-ACOPF) problem.
- Here is the document link: https://gocompetition.energy.gov/sites/default/files/Challenge3_Problem_Formulation_20230126.pdf
- 53 Pages, 163 equations, hundreds of notations!

Grid Optimization Competition
Challenge 3 Problem Formulation

Jesse Holzer Carleton Coffrin Christopher DeMarco Ray Duthu
Stephen Elbert Brent Eldridge Tarek Elgindy Scott Greene
Nongchao Guo Elaine Hale Bernard Lesieutre Terrence Mak
Colin McMillan Hans Mittelmann Hyungseon Oh
Richard O'Neill Thomas Overbye Bryan Palmintier
Farnaz Safdarian Ahmad Tbaileh Pascal Van Hentenryck
Arun Veeramany Jessica Wert

January 26, 2023

Contents

Comprehensive Grid Optimization Model

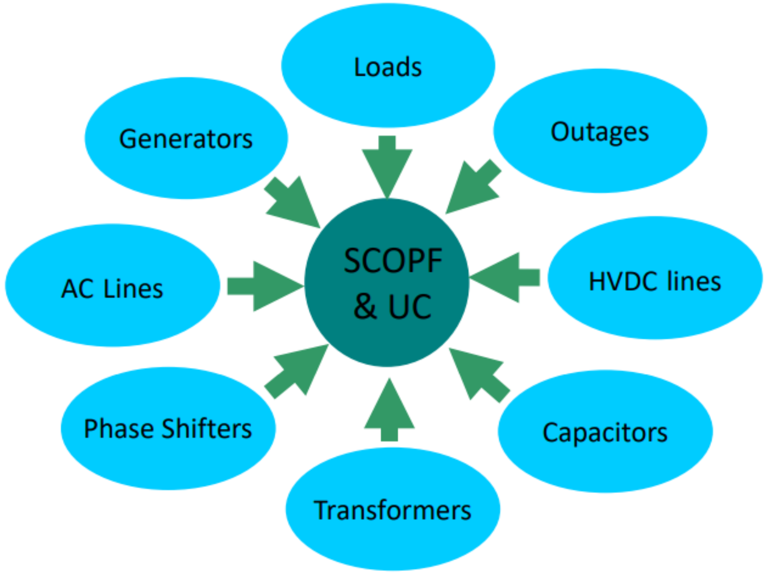
GO3 Problems

GO3: Multi-period AC Optimal Power Flow with Unit Commitment and Line Switching (2022)

- **Market Surplus Objective (for D1, D2, D3)**
  - Maximize:

    Total Market Surplus = Base Case Market Surplus + Worst Case and Average Case of Post-Contingency Outcomes.
- **Constraints**
  - Bus real and reactive power balance and voltage limits
  - Zonal reserve requirements
  - Device on-off status and related constraints
  - Producing and consuming devices: startup, shutdown, dispatchable power, ramping, reserve, max/min energy over multiple intervals
  - Shunt real and reactive power constraints
  - AC & DC branch flow limits
  - Post-contingency AC power flow limits

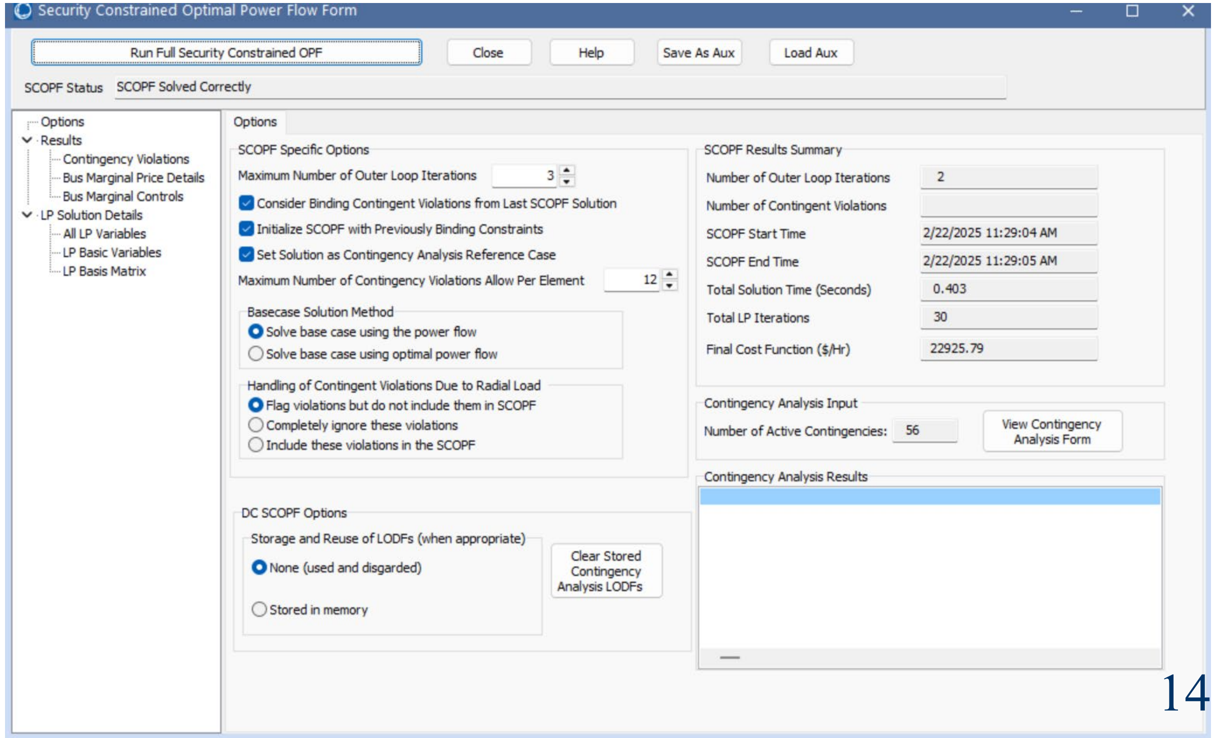
- D1: Real-Time Market with 8-hour look ahead -- 8 0.25-hour periods, 8 0.5-hour periods, 2 1-hour periods
- D2: Day-Ahead Market with 48-hour look ahead -- 48 1-hour periods
- D3: Weekly Scheduling Week-Ahead Advisory with 7-day (168-hour) look ahead -- 42 4-hour periods



Source: https://gocompetition.energy.gov/sites/default/files/YongOptimization_Challenge3_Presentation_20231114.pdf

PowerWorld SCOPF Application

- To see the PowerWorld SCOPF application, first open the AGC37_SCOPF case and set the load multiplier to 0.9 and solve the case with the OPF; look at the results.
- Then select **Tools, Contingency Analysis** to verify that some contingencies have been defined
  - On the **Contingency Analysis** form click **Start Run** to do the contingency analysis; note the violations
- Select **Add Ons, SCOPF** to open the SCOPF.
- Click **Run Full Security Constrained OPF.**



SCOPF Status: SCOPF Solved Correctly

SCOPF Specific Options

Maximum Number of Outer Loop Iterations: 3

☑ Consider Binding Contingent Violations from Last SCOPF Solution

☑ Initialize SCOPF with Previously Binding Constraints

☑ Set Solution as Contingency Analysis Reference Case

Maximum Number of Contingency Violations Allow Per Element: 12

Basecase Solution Method
● Solve base case using the power flow
○ Solve base case using optimal power flow

Handling of Contingent Violations Due to Radial Load
● Flag violations but do not include them in SCOPF
○ Completely ignore these violations
○ Include these violations in the SCOPF

DC SCOPF Options

Storage and Reuse of LODFs (when appropriate)
● None (used and disgarded)
○ Stored in memory

SCOPF Results Summary

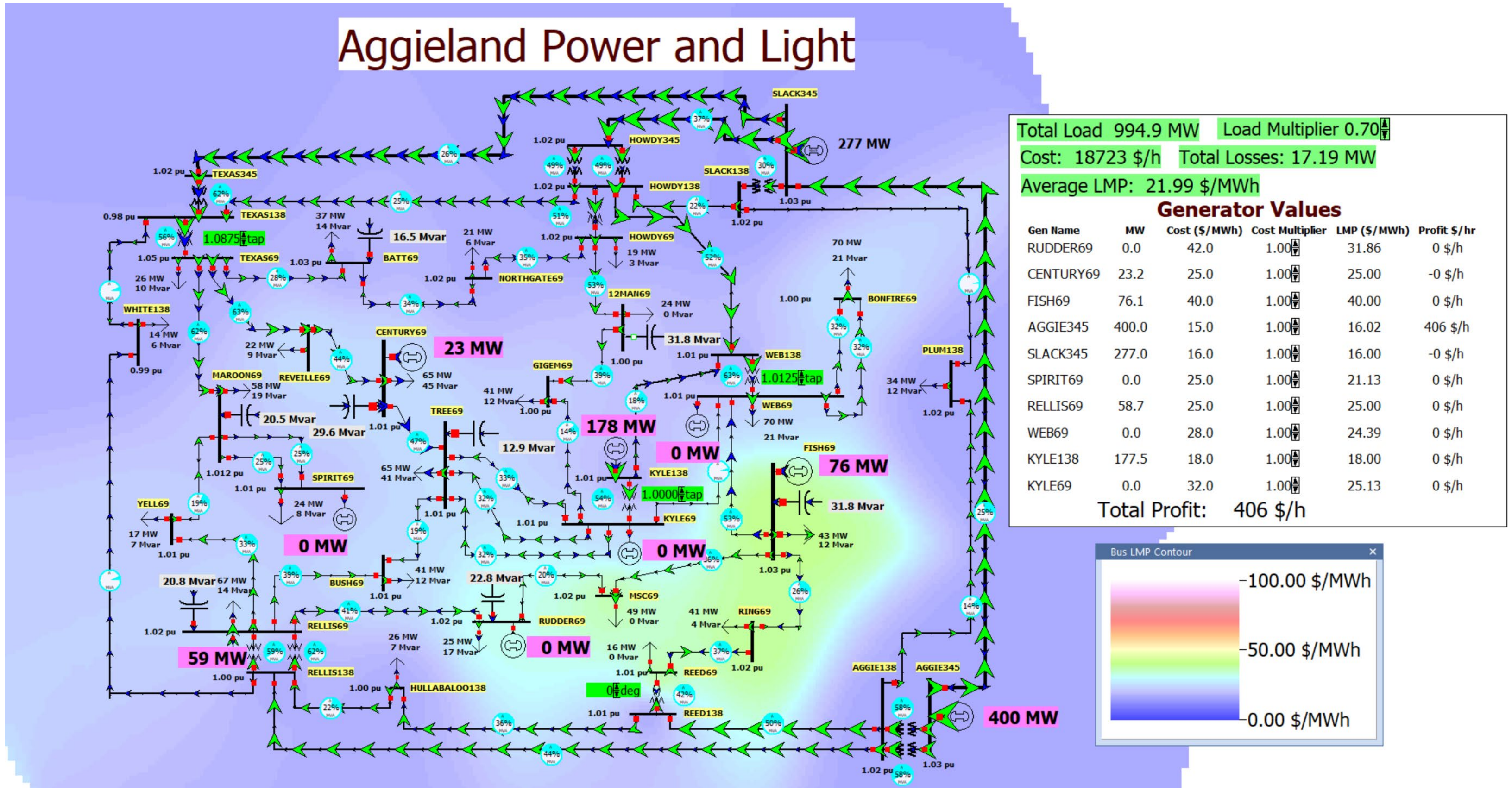
Number of Outer Loop Iterations	2
Number of Contingent Violations	
SCOPF Start Time	2/22/2025 11:29:04 AM
SCOPF End Time	2/22/2025 11:29:05 AM
Total Solution Time (Seconds)	0.403
Total LP Iterations	30
Final Cost Function ($/Hr)	22925.79

Contingency Analysis Input

Number of Active Contingencies: 56

37 Bus Case SCOPF Results

- Keeping the SCOPF form open, contour the bus LMPs.
- What had been a relatively boring OPF solution indicates some major issues.
- Looking at the SCOPF form **Results, Contingency Violations** indicates there are some contingencies with unenforceable constraints.



Total Load 994.9 MW Load Multiplier 0.70

Cost: 18723 $/h Total Losses: 17.19 MW

Average LMP: 21.99 $/MWh

Generator Values

Gen Name	MW	Cost ($/MWh)	Cost Multiplier	LMP ($/MWh)	Profit $/hr
RUDDER69	0.0	42.0	1.00	31.86	0 $/h
CENTURY69	23.2	25.0	1.00	25.00	-0 $/h
FISH69	76.1	40.0	1.00	40.00	0 $/h
AGGIE345	400.0	15.0	1.00	16.02	406 $/h
SLACK345	277.0	16.0	1.00	16.00	-0 $/h
SPIRIT69	0.0	25.0	1.00	21.13	0 $/h
RELLIS69	58.7	25.0	1.00	25.00	0 $/h
WEB69	0.0	28.0	1.00	24.39	0 $/h
KYLE138	177.5	18.0	1.00	18.00	0 $/h
KYLE69	0.0	32.0	1.00	25.13	0 $/h

Total Profit: 406 $/h

LP OPF and SCOPF Issues

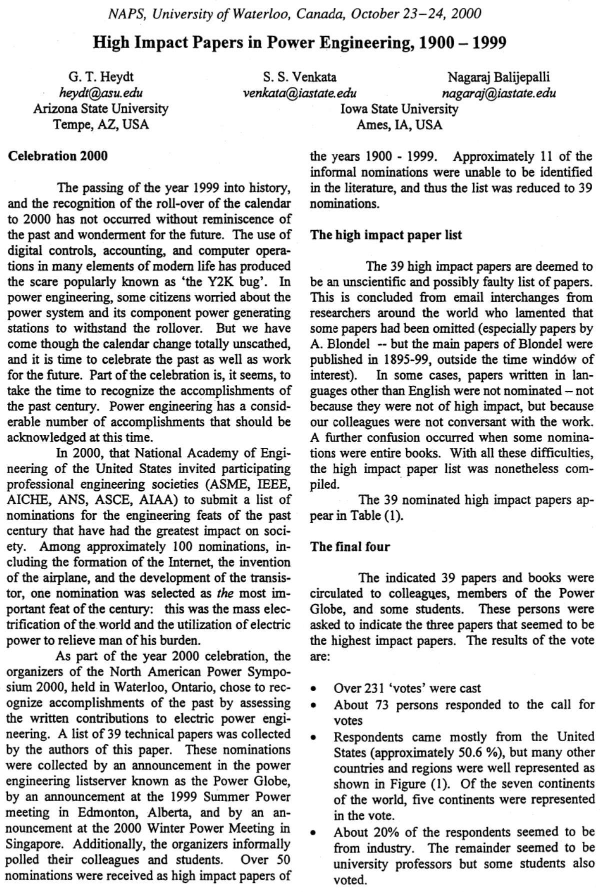
- The LP approach is widely used for the OPF and SCOPF, particularly when implementing a dc power flow approach.
- A key issue is determining the number of binding constraints to enforce in the LP tableau.
  - Enforcing too many is time-consuming, enforcing too few results in excessive iterations
- The LP approach is limited by the degree of linearity in the power system.
  - Real power constraints are fairly linear, reactive power constraints much less so

Most Impactful Electric Power Systems Papers

- At TPEC 2025 Jerry Heydt, Line Roald, Kate Davis and I presented a paper that provides results of a survey seeking to identify the most impactful power system papers written in English between 1975 and 2024.
- We had two goals in writing the paper:
    1. To benefit history by determining the papers from 1975-2024 that people viewed as the mort important at the end of the first quarter of the 21st Century
    2. To provide guidance to future authors on writing impactful papers by providing a listing of some of the most impactful recent ones
- The paper also provides a nice summary of the state-of-the-art in 1975.
    - Many of the key technologies and algorithms in use today actually were known before 1975; examples include power flow, optimal power flow, state estimation, electromagnetic transients, stochastic power flow, interactive computing, prototype energy management systems, digital relays, and much more

Building on a Survey from 2000

- This paper is a follow up to one from NAPS 2000 that identified the most impactful power system papers of the 20th Century.



NAPS, University of Waterloo, Canada, October 23–24, 2000

High Impact Papers in Power Engineering, 1900 – 1999

G. T. Heydt
heydt@asu.edu
Arizona State University
Tempe, AZ, USA

S. S. Venkata
venkata@iastate.edu

Nagaraj Balijepalli
nagaraj@iastate.edu

Iowa State University
Ames, IA, USA

Celebration 2000

The passing of the year 1999 into history, and the recognition of the roll-over of the calendar to 2000 has not occurred without reminiscence of the past and wonderment for the future. The use of digital controls, accounting, and computer operations in many elements of modern life has produced the scare popularly known as 'the Y2K bug'. In power engineering, some citizens worried about the power system and its component power generating stations to withstand the rollover. But we have come though the calendar change totally unscathed, and it is time to celebrate the past as well as work for the future. Part of the celebration is, it seems, to take the time to recognize the accomplishments of the past century. Power engineering has a considerable number of accomplishments that should be acknowledged at this time.

In 2000, that National Academy of Engineering of the United States invited participating professional engineering societies (ASME, IEEE, AICHE, ANS, ASCE, AIAA) to submit a list of nominations for the engineering feats of the past century that have had the greatest impact on society. Among approximately 100 nominations, including the formation of the Internet, the invention of the airplane, and the development of the transistor, one nomination was selected as *the* most important feat of the century: this was the mass electrification of the world and the utilization of electric power to relieve man of his burden.

As part of the year 2000 celebration, the organizers of the North American Power Symposium 2000, held in Waterloo, Ontario, chose to recognize accomplishments of the past by assessing the written contributions to electric power engineering. A list of 39 technical papers was collected by the authors of this paper. These nominations were collected by an announcement in the power engineering listserver known as the Power Globe, by an announcement at the 1999 Summer Power meeting in Edmonton, Alberta, and by an announcement at the 2000 Winter Power Meeting in Singapore. Additionally, the organizers informally polled their colleagues and students. Over 50 nominations were received as high impact papers of the years 1900 - 1999. Approximately 11 of the informal nominations were unable to be identified in the literature, and thus the list was reduced to 39 nominations.

The high impact paper list

The 39 high impact papers are deemed to be an unscientific and possibly faulty list of papers. This is concluded from email interchanges from researchers around the world who lamented that some papers had been omitted (especially papers by A. Blondel -- but the main papers of Blondel were published in 1895-99, outside the time window of interest). In some cases, papers written in languages other than English were not nominated – not because they were not of high impact, but because our colleagues were not conversant with the work. A further confusion occurred when some nominations were entire books. With all these difficulties, the high impact paper list was nonetheless compiled.

The 39 nominated high impact papers appear in Table (1).

The final four

The indicated 39 papers and books were circulated to colleagues, members of the Power Globe, and some students. These persons were asked to indicate the three papers that seemed to be the highest impact papers. The results of the vote are:

- Over 231 'votes' were cast
- About 73 persons responded to the call for votes
- Respondents came mostly from the United States (approximately 50.6 %), but many other countries and regions were well represented as shown in Figure (1). Of the seven continents of the world, five continents were represented in the vote.
- About 20% of the respondents seemed to be from industry. The remainder seemed to be university professors but some students also voted.

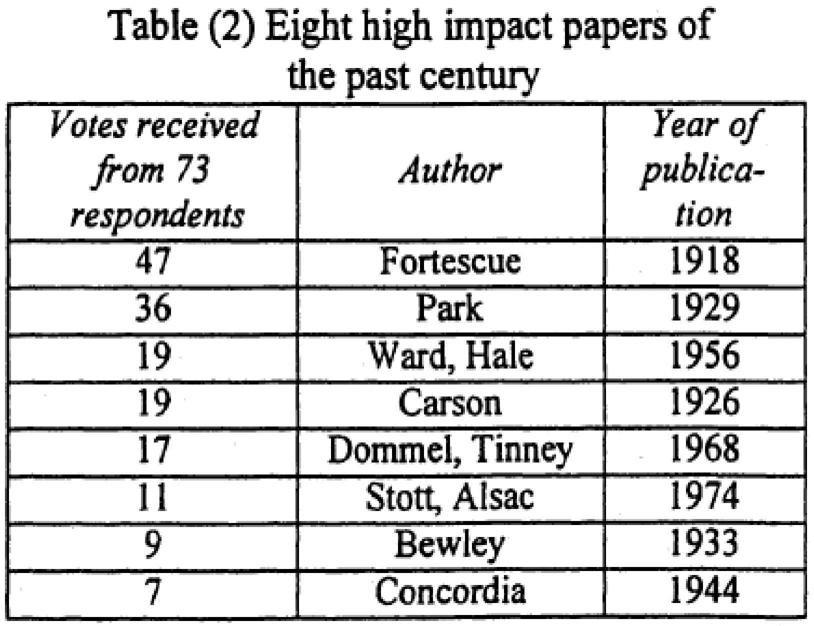
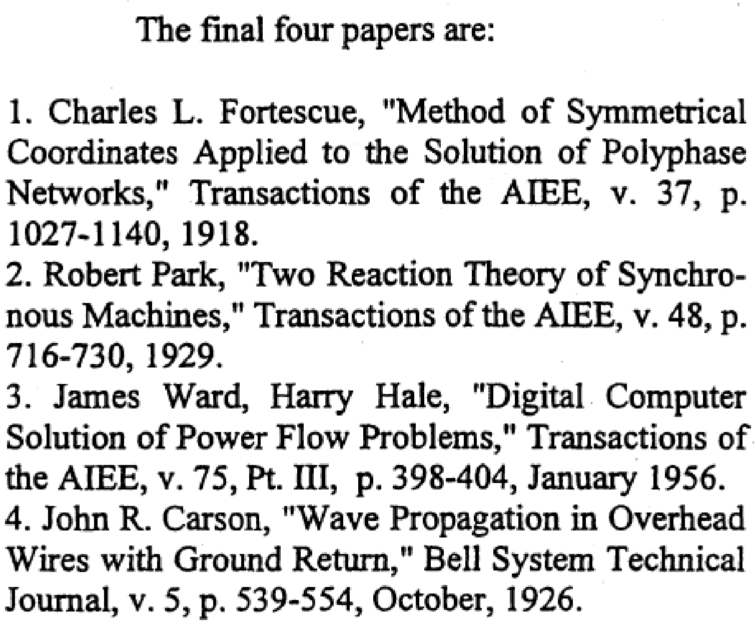


Table (2) Eight high impact papers of the past century

Votes received from 73 respondents	*Author*	*Year of publication*
47	Fortescue	1918
36	Park	1929
19	Ward, Hale	1956
19	Carson	1926
17	Dommel, Tinney	1968
11	Stott, Alsac	1974
9	Bewley	1933
7	Concordia	1944



The final four papers are:

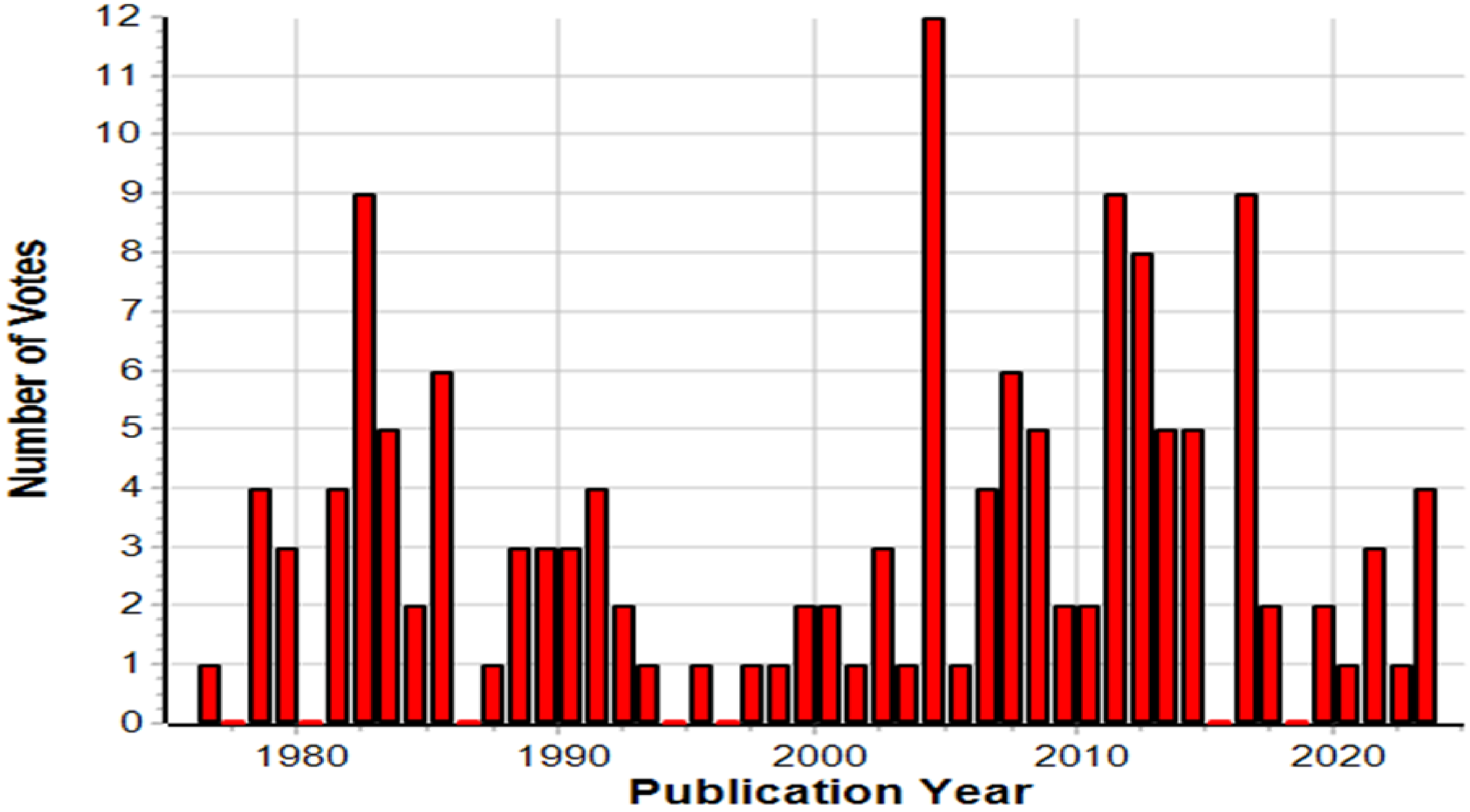
1. Charles L. Fortescue, "Method of Symmetrical Coordinates Applied to the Solution of Polyphase Networks," Transactions of the AIEE, v. 37, p. 1027-1140, 1918.
2. Robert Park, "Two Reaction Theory of Synchronous Machines," Transactions of the AIEE, v. 48, p. 716-730, 1929.
3. James Ward, Harry Hale, "Digital Computer Solution of Power Flow Problems," Transactions of the AIEE, v. 75, Pt. III, p. 398-404, January 1956.
4. John R. Carson, "Wave Propagation in Overhead Wires with Ground Return," Bell System Technical Journal, v. 5, p. 539-554, October, 1926.

Our Approach

- In late 2024 we surveyed members of the power system community asking for up to three papers that they viewed as most impactful published between 1975 and 2024.
- We received 153 papers from 76 different people (a relatively small number of responses, but comparable to the 2000 survey).
- Like any survey, the results are biased by who responds; here the responses were primarily from electric power system researchers, as opposed to practicing engineers.
  - The survey was widely distributed (more than 10,000 people), but many of the recipients probably did not have an in-depth knowledge of the technical literature
  - The survey might have included some less impactful papers, and might have missed some others, but overall we think it correctly identified the most impactful papers

Votes by Publication Year

- Almost all the responses were for papers from IEEE Power and Energy Society (PES) Journals, and many were from the last 20 years.



The Top Papers with Most in 615 Areas!

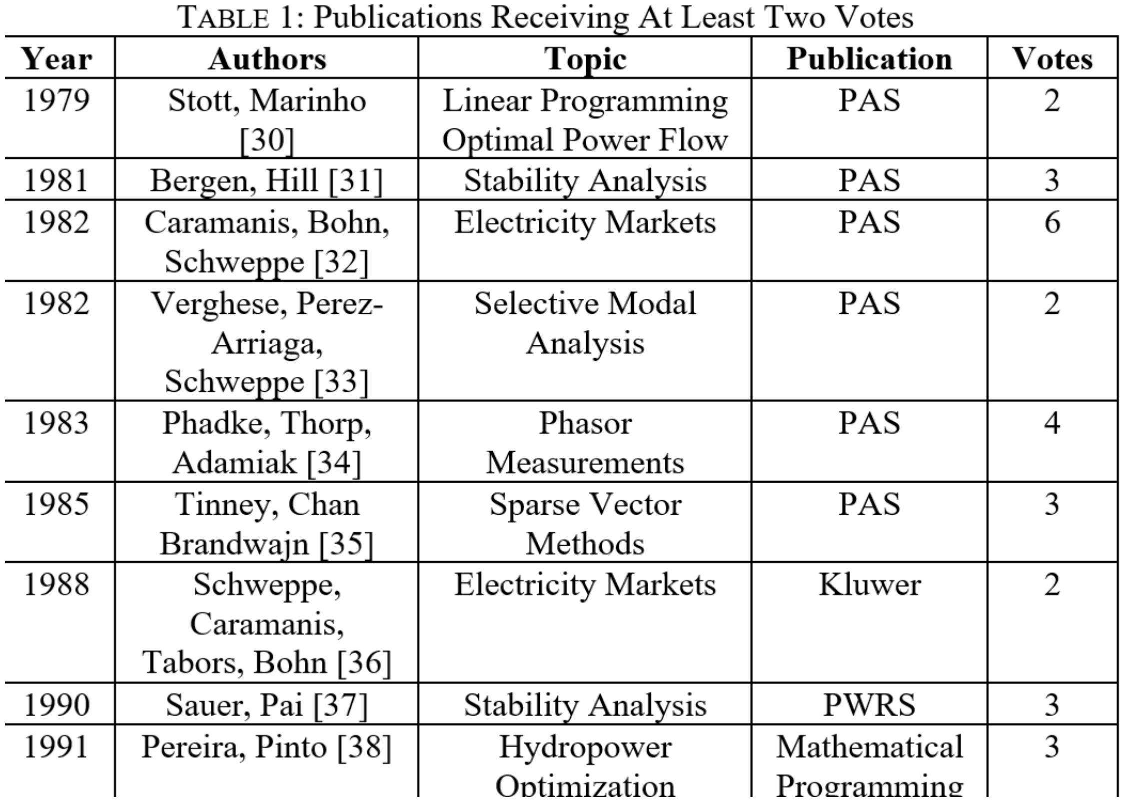
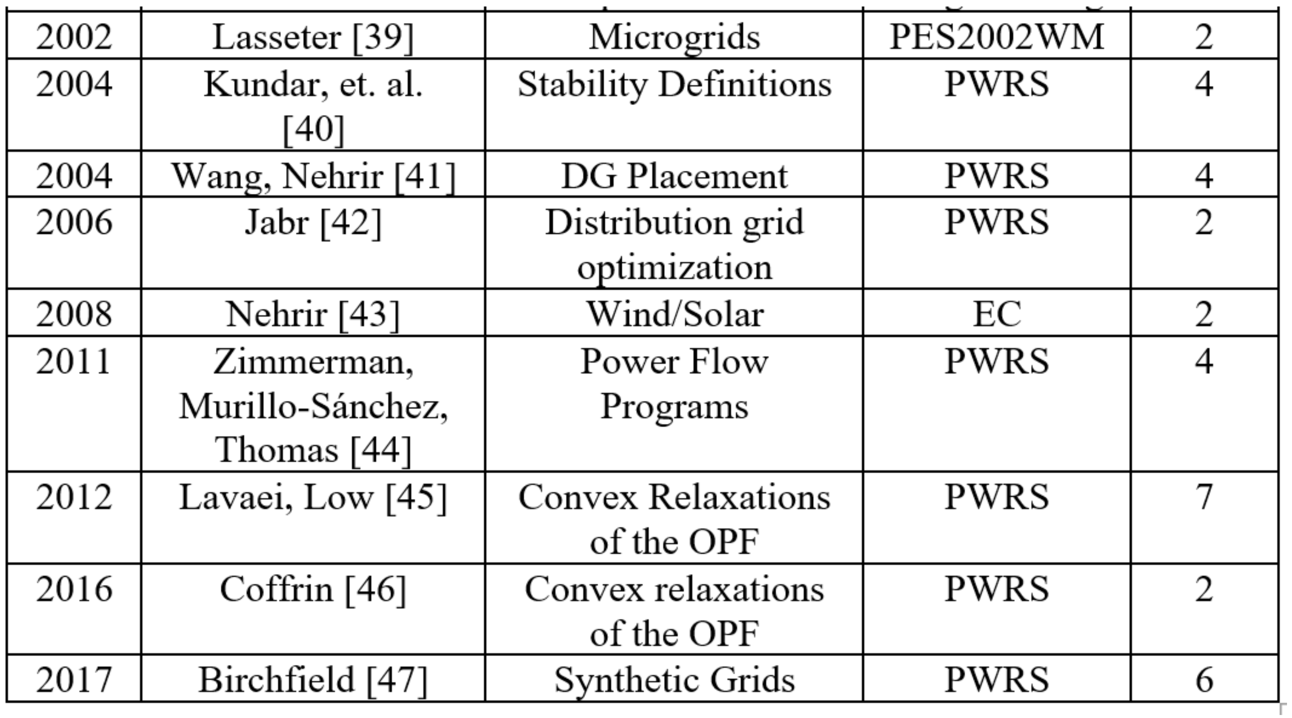


TABLE 1: Publications Receiving At Least Two Votes



Year	Authors	Topic	Publication	Votes
1979	Stott, Marinho [30]	Linear Programming Optimal Power Flow	PAS	2
1981	Bergen, Hill [31]	Stability Analysis	PAS	3
1982	Caramanis, Bohn, Schweppe [32]	Electricity Markets	PAS	6
1982	Verghese, Perez-Arriaga, Schweppe [33]	Selective Modal Analysis	PAS	2
1983	Phadke, Thorp, Adamiak [34]	Phasor Measurements	PAS	4
1985	Tinney, Chan Brandwajn [35]	Sparse Vector Methods	PAS	3
1988	Schweppe, Caramanis, Tabors, Bohn [36]	Electricity Markets	Kluwer	2
1990	Sauer, Pai [37]	Stability Analysis	PWRS	3
1991	Pereira, Pinto [38]	Hydropower Optimization	Mathematical Programming	3
2002	Lasseter [39]	Microgrids	PES2002WM	2
2004	Kundar, et. al. [40]	Stability Definitions	PWRS	4
2004	Wang, Nehrir [41]	DG Placement	PWRS	4
2006	Jabr [42]	Distribution grid optimization	PWRS	2
2008	Nehrir [43]	Wind/Solar	EC	2
2011	Zimmerman, Murillo-Sánchez, Thomas [44]	Power Flow Programs	PWRS	4
2012	Lavaei, Low [45]	Convex Relaxations of the OPF	PWRS	7
2016	Coffrin [46]	Convex relaxations of the OPF	PWRS	2
2017	Birchfield [47]	Synthetic Grids	PWRS	6

Some Comments

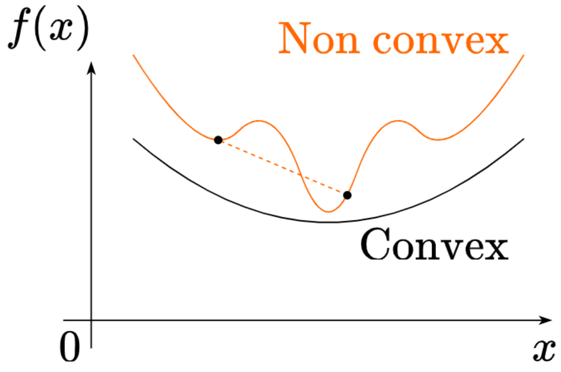
- A major theme was the breadth of the received responses. Of the 144 votes most of the nominated papers only received a single vote.
- Some of this could be due to the sheer volume of papers now being published.
  - In I975 PES has one main journal publication with a page count of about 2200, an number that had grown to 3700 in 1985
  - In 2023 PES has seven journal publications with a total page count of more than 22,000!
  - There are also now many conferences, including the Texas Power and Energy Conference!
- With so many papers being published, it is hard for any single one to stand out!

AC OPF Convex Relaxations

- Optimization problems can also be divided into convex optimization and nonconvex optimization.
- Convex function: a function f(x) is called convex if and only if

For all $0 \leq t \leq 1$ and all $x_1, x_2 \in X$:

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$



- **Convex optimization**: the objective function $f(x)$ and inequality constraint function $g_i(x)$ are all convex functions, and equality constraint function $h_j(x)$ are all linear functions, i.e., $h_j(x) = a_j^\top x + b_j$.
- In general, the ac OPF does not have this form. Convex relaxations of the power flow equations can make the problem easier but still is good results.

Two Papers in this Category

92 IEEE TRANSACTIONS ON POWER SYSTEMS, VOL. 27, NO. 1, FEBRUARY 2012

Zero Duality Gap in Optimal Power Flow Problem

Javad Lavaei, *Senior Member, IEEE*, and Steven H. Low, *Fellow, IEEE*

***Abstract*—The optimal power flow (OPF) problem is nonconvex and generally hard to solve. In this paper, we propose a semidefinite programming (SDP) optimization, which is the dual of an equivalent form of the OPF problem. A global optimum solution to the OPF problem can be retrieved from a solution of this convex dual problem whenever the duality gap is zero. A necessary and sufficient condition is provided in this paper to guarantee the existence of no duality gap for the OPF problem. This condition is satisfied by the standard IEEE benchmark systems with 14, 30, 57, 118, and 300 buses as well as several randomly generated systems. Since this condition is hard to study, a sufficient zero-duality-gap condition is also derived. This sufficient condition holds for IEEE systems after small resistance (10^{-5} per unit) is added to every transformer that originally assumes zero resistance. We investigate this sufficient condition and justify that it holds widely in practice. The main underlying reason for the successful convexification of the OPF problem can be traced back to the modeling of transformers and transmission lines as well as the non-negativity of physical quantities such as resistance and inductance.**

***Index Terms*—Convex optimization, linear matrix inequality, optimal power flow, polynomial-time algorithm, power system.**

I. INTRODUCTION

THE optimal power flow (OPF) problem deals with finding an optimal operating point of a power system that minimizes an appropriate cost function such as generation cost or transmission loss subject to certain constraints on power and voltage variables [1]. Started by the work [2] in 1962, the OPF problem has been extensively studied in the literature and numerous algorithms have been proposed for solving this highly nonconvex problem [3]–[5], including linear programming, Newton-Raphson, quadratic programming, nonlinear programming, Lagrange relaxation, interior point methods, artificial intelligence, artificial neural network, fuzzy logic, genetic algorithm, evolutionary programming, and particle swarm optimization [1], [6]–[8]. A good number of these methods are based on the Karush-Kuhn-Tucker (KKT) necessary conditions, which can only guarantee a locally optimal solution, in light of the nonconvexity of the OPF problem [9]. This nonconvexity is partially due to the nonlinearity of physical parameters, namely active power, reactive power, and voltage magnitude.

In the past decade, much attention has been paid to devising efficient algorithms with guaranteed performance for the OPF problem. For instance, the recent papers [10] and [11] propose nonlinear interior-point algorithms for an equivalent current injection model of the problem. An improved implementation of the automatic differentiation technique for the OPF problem is studied in the recent work [12]. In an effort to convexify the OPF problem, it is justified in [13] that the load flow problem of a radial distribution system may be modeled as a convex optimization problem in the form of a conic program. Nonetheless, the results fail to hold for a meshed network, due to the presence of arctangent equality constraints [14]. Nonconvexity appears in more sophisticated power problems such as the stability constrained OPF problem where the stability at the operating point is an extra constraint [15], [16] or the dynamic OPF problem where the dynamics of the generators are also taken into account [17], [18]. The recent paper [19] proposes a convex relaxation to solve the OPF problem efficiently and tests its results on IEEE systems. Some of the results derived in the present work are related to this well-known convex relaxation. However, [19] drops a rank constraint in the original OPF without any justification in order to obtain a convex formulation.

As will be shown in this paper, the OPF problem is NP-hard in the worst case. Our recent work also proves that a closely related problem of finding an optimal operating point of a radiating antenna circuit is an NP-complete problem, by reducing the number partitioning problem to the antenna problem [20]. The goal of the present work is to exploit the physical properties of power systems and obtain a polynomial-time algorithm to find a global optimum of the OPF problem for a large class of power networks.

In this paper, we suggest solving the dual of an equivalent form of the OPF problem (referred to as the dual OPF problem), rather than the OPF problem itself. This dual problem is a convex semidefinite program and therefore can be solved efficiently (in polynomial time). However, the optimal objective value of the dual problem is only a lower bound on the optimal value of the original OPF problem and the lower bound may not be tight (in presence of a nonzero duality gap) [21]. A globally optimal solution to the OPF problem can be recovered from a solution to the dual OPF problem if the duality gap is zero (i.e., strongly duality holds between these two optimizations). In this paper, we derive a necessary and sufficient condition to guarantee zero duality gap. Interestingly, this condition is satisfied for all the five IEEE benchmark systems archived at [22] with 14, 30, 57, 118, and 300 buses, in addition to several randomly generated systems. In other words, these practical systems can all be convexified via the new formulation proposed here. In

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J. Lavaei is with the Department of Control and Dynamical Systems, California Institute of Technology, Pasadena, CA 91125 USA (e-mail: lavaei@cds.caltech.edu).

S. H. Low is with Computer Science and Electrical Engineering Departments, California Institute of Technology, Pasadena, CA 91125 USA (e-mail: slow@caltech.edu).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

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3008 IEEE TRANSACTIONS ON POWER SYSTEMS, VOL. 31, NO. 4, JULY 2016

The QC Relaxation: A Theoretical and Computational Study on Optimal Power Flow

Carleton Coffrin, Hassan L. Hijazi, and Pascal Van Hentenryck

***Abstract*—Convex relaxations of the power flow equations and, in particular, the semi-definite programming (SDP) and second-order cone (SOC) relaxations, have attracted significant interest in recent years. The quadratic convex (QC) relaxation is a departure from these relaxations in the sense that it imposes constraints to preserve stronger links between the voltage variables through convex envelopes of the polar representation. This paper is a systematic study of the QC relaxation for AC optimal power flow with realistic side constraints. The main theoretical result shows that the QC relaxation is stronger than the SOC relaxation and neither dominates nor is dominated by the SDP relaxation. In addition, comprehensive computational results show that the QC relaxation may produce significant improvements in accuracy over the SOC relaxation at a reasonable computational cost, especially for networks with tight bounds on phase angle differences. The QC and SOC relaxations are also shown to be significantly faster and reliable compared to the SDP relaxation given the current state of the respective solvers.**

***Index Terms*—Convex quadratic optimization, optimal power flow, optimization methods.**

NOMENCLATURE

N	Set of nodes in the network.
E	Set of *from* edges in the network.
E^R	Set of *to* edges in the network.
$\boldsymbol{i}$	Imaginary number constant.
I	AC current.
$S = p + \boldsymbol{i}q$	AC power.
$V = v\angle\theta$	AC voltage.
$Z = r + \boldsymbol{i}x$	Line impedance.
$Y = g + \boldsymbol{i}b$	Line admittance.
$T = t\angle\theta^t$	Transformer properties.
$Y^s = g^s + \boldsymbol{i}b^s$	Bus shunt admittance.
W	Product of two AC voltages.
l	Current magnitude squared, $\lvert I\rvert^2$.
b^c	Line charging.
s^u	Line apparent power thermal limit.
θ^Δ	Phase angle difference limit.
$S^d = p^d + \boldsymbol{i}q^d$	AC power demand.
$S^g = p^g + \boldsymbol{i}q^g$	AC power generation.
c_0, c_1, c_2	Generation cost coefficients.
$\Re(\cdot)$	Real part of a complex number.
$\Im(\cdot)$	Imaginary part of a complex number.
$(\cdot)^*$	Conjugate of a complex number.
$\lvert\cdot\rvert$	Magnitude of a complex number, l^2-norm.
x^l, x^u	Lower and upper bounds of x, respectively.
$\breve{x}$	Convex envelope of x.
$\boldsymbol{x}$	Constant value.

I. INTRODUCTION

CONVEX relaxations of the power flow equations have attracted significant interest in recent years. They include the semi-definite programming (SDP) [1], second-order cone (SOC) [2], Convex-DistFlow (CDF) [3], and the recent quadratic convex (QC) [4] and moment-based [5], [6] relaxations. Much of the excitement underlying this line of research comes from the fact that the SDP relaxation has shown to be tight on a variety of case studies [7], opening a new avenue for accurate, reliable, and efficient solutions to a variety of power system applications. Indeed, industrial-strength optimization tools (e.g., Gurobi, cplex, Mosek) are now available to solve various classes of convex optimization problems.

The relationships between the SDP, SOC, and CDF relaxations is now largely well-understood: see [8], [9] for a comprehensive overview. In particular, the SOC and CDF relaxations are known to be equivalent and the SDP relaxation is at least as strong as both of these. However, little is known about the QC relaxation which is a significant departure from these more traditional relaxations. Indeed, one of the key features of the QC relaxation is to compute convex envelopes of the polar representation of the power flow equations in the hope of preserving stronger links between the voltage variables. This contrasts with the SDP and SOC relaxations which are derived from a lift-and-project approach on the complex representation.

This paper fills this gap and provides a theoretical study of the QC relaxation as well as a comprehensive computational evaluation to compare the strengths and weaknesses of these relaxations. Our main contributions can be summarized as follows:

1) The QC relaxation is stronger than the SOC relaxation.
2) The QC relaxation neither dominates nor is dominated by the SDP relaxation.

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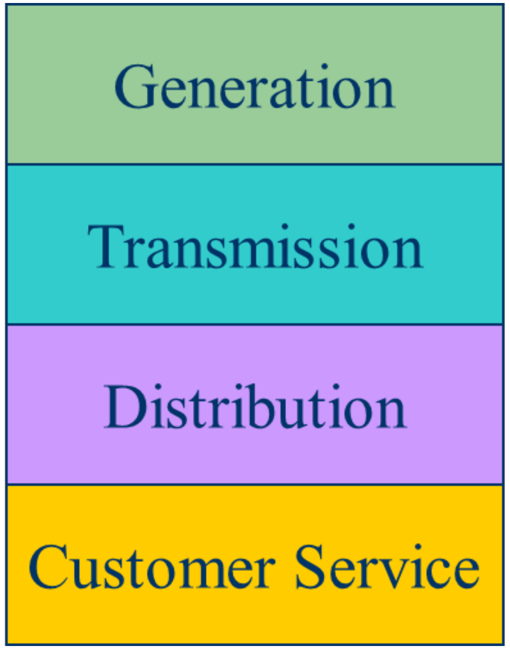
The authors are with the Optimisation Research Group, NICTA, ACT 2601, Australia, and also with the College of Engineering and Computer Science, Australian National University, ACT 0200, Australia.

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TPWRS.2015.2463111

Electricity Markets History

- For decades electric utilities operated as vertical monopolies, with their rates set by state regulators.
- Utilities had an obligation to serve and customers had no choice.
  - There was little third-party generation
- Major change in US occurred in 1992 with the National Energy Policy Act that mandated utilities provide "nondiscriminatory" access to the high voltage grid.
- Goal was to setup true competition in generation.



Markets Versus Centralized Planning

- With the vertically integrated utility, a small number of entities (typically utilities) did most of the planning.
  - For example, which new generators and/or lines to build
  - Planning was coordinated and governed by regulators
  - Regulators needed to know the utilities actual costs so they could provide them with a fixed rate of return
- With markets the larger number of participants often make individual decisions in reaction to prices.
  - For example, whether to build new generation
  - Generator owners in general to not need to reveal their true costs; rather they make offers into the market

Overall Goal

- Goal is to maximize the economic surplus (or total welfare), which is the sum of the consumer surplus and the producer surplus (i.e., their profit).
- Generation owners have to decide their offer prices.
- If their price is too high, they are not selected to generate.
- At the wholesale level, the consumers often just see a price, though there can be price responsive load bids.

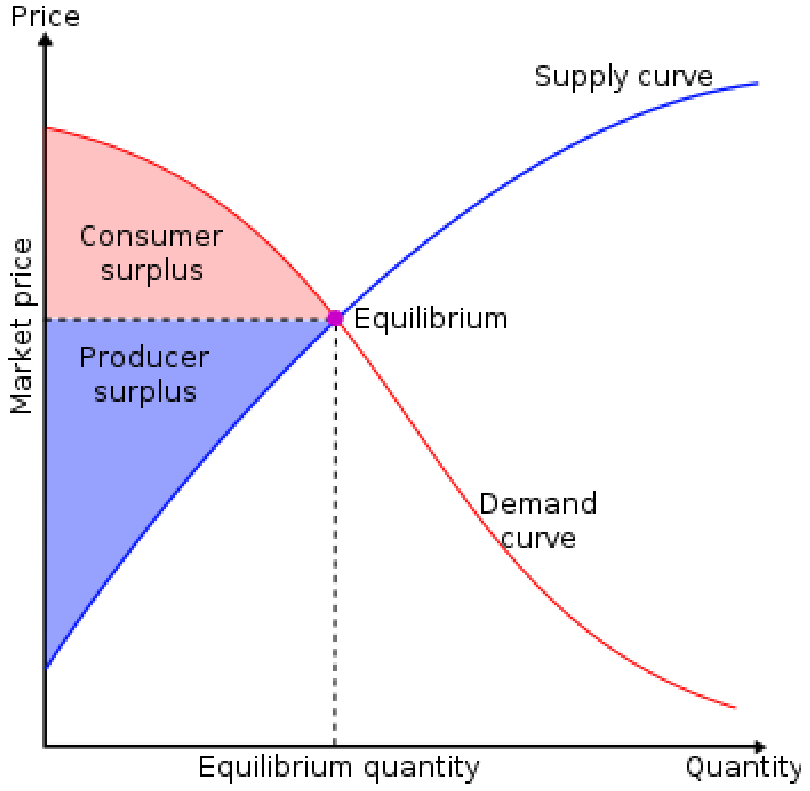


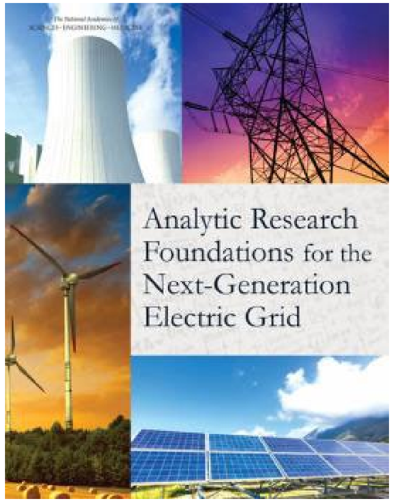
Image Source: en.wikipedia.org/wiki/Economic_surplus#/media/File:Economic-surpluses.svg

Electricity Market History

- Power pools have been used for almost 100 years, in which utilities created agreements to buy and sell electricity with their neighbors.
  - PJM (originally Pennsylvania-New Jersey-Maryland) formed in 1927
- The methodology used to determine the price was the production cost; each utility calculated how much it would cost them to produce more power (sell price), or how much they would say if they produced less (buy price); if the sell price for one utility was less than the buy price for another, then they would transact, usually splitting the savings.
- In the 1990's there was a goal of creating more flexible electricity markets.

Multidisciplinary Research in Power and Economics

- The development of true power markets required collaboration between power engineers and economists with the nice description of how some of this developed within the Power Systems Engineering Research Center (PSERC, with Texas A&M a member) described in Chapter 8 of [1].
  - One of the challenges was agreeing on notation, with the power engineers treating P and Q as power, and the economics as price and quanity



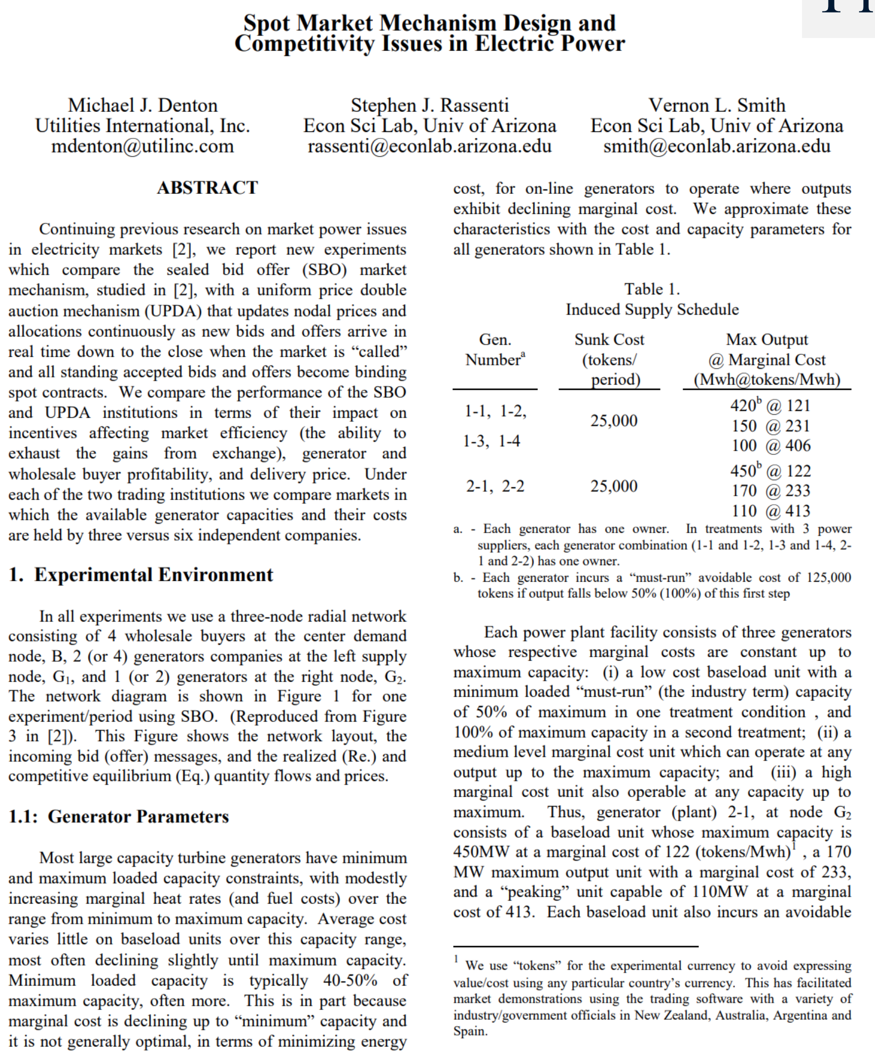
Analytic Research Foundations for the Next-Generation Electric Grid

- The Hawaiian International Conference on Science Sciences (HICSS) also played a major role, with one of the participants winning the Nobel Price for Economics in 2002 (Vernon Smith) (he was a TAMU Hagler Fellow in class of 2012-2013).

[1] US National Academies, Analytic *Research Foundations for the Next-Generation Electric Grid*, 2016

Example Vernon Smith Paper from HICSS (1998)

First two pages



Spot Market Mechanism Design and Competitivity Issues in Electric Power

Michael J. Denton
Utilities International, Inc.
mdenton@utilinc.com

Stephen J. Rassenti
Econ Sci Lab, Univ of Arizona
rassenti@econlab.arizona.edu

Vernon L. Smith
Econ Sci Lab, Univ of Arizona
smith@econlab.arizona.edu

ABSTRACT

Continuing previous research on market power issues in electricity markets [2], we report new experiments which compare the sealed bid offer (SBO) market mechanism, studied in [2], with a uniform price double auction mechanism (UPDA) that updates nodal prices and allocations continuously as new bids and offers arrive in real time down to the close when the market is “called” and all standing accepted bids and offers become binding spot contracts. We compare the performance of the SBO and UPDA institutions in terms of their impact on incentives affecting market efficiency (the ability to exhaust the gains from exchange), generator and wholesale buyer profitability, and delivery price. Under each of the two trading institutions we compare markets in which the available generator capacities and their costs are held by three versus six independent companies.

1. Experimental Environment

In all experiments we use a three-node radial network consisting of 4 wholesale buyers at the center demand node, B, 2 (or 4) generators companies at the left supply node, G_1, and 1 (or 2) generators at the right node, G_2. The network diagram is shown in Figure 1 for one experiment/period using SBO. (Reproduced from Figure 3 in [2]). This Figure shows the network layout, the incoming bid (offer) messages, and the realized (Re.) and competitive equilibrium (Eq.) quantity flows and prices.

1.1: Generator Parameters

Most large capacity turbine generators have minimum and maximum loaded capacity constraints, with modestly increasing marginal heat rates (and fuel costs) over the range from minimum to maximum capacity. Average cost varies little on baseload units over this capacity range, most often declining slightly until maximum capacity. Minimum loaded capacity is typically 40-50% of maximum capacity, often more. This is in part because marginal cost is declining up to “minimum” capacity and it is not generally optimal, in terms of minimizing energy cost, for on-line generators to operate where outputs exhibit declining marginal cost. We approximate these characteristics with the cost and capacity parameters for all generators shown in Table 1.

Table 1.
Induced Supply Schedule

Gen. Number[a]	Sunk Cost (tokens/ period)	Max Output @ Marginal Cost (Mwh@tokens/Mwh)
1-1, 1-2, 1-3, 1-4	25,000	420[b] @ 121 150 @ 231 100 @ 406
2-1, 2-2	25,000	450[b] @ 122 170 @ 233 110 @ 413

a. - Each generator has one owner. In treatments with 3 power suppliers, each generator combination (1-1 and 1-2, 1-3 and 1-4, 2-1 and 2-2) has one owner.

b. - Each generator incurs a “must-run” avoidable cost of 125,000 tokens if output falls below 50% (100%) of this first step

Each power plant facility consists of three generators whose respective marginal costs are constant up to maximum capacity: (i) a low cost baseload unit with a minimum loaded “must-run” (the industry term) capacity of 50% of maximum in one treatment condition , and 100% of maximum capacity in a second treatment; (ii) a medium level marginal cost unit which can operate at any output up to the maximum capacity; and (iii) a high marginal cost unit also operable at any capacity up to maximum. Thus, generator (plant) 2-1, at node G_2 consists of a baseload unit whose maximum capacity is 450MW at a marginal cost of 122 (tokens/Mwh)[1] , a 170 MW maximum output unit with a marginal cost of 233, and a “peaking” unit capable of 110MW at a marginal cost of 413. Each baseload unit also incurs an avoidable

---

[1] We use “tokens” for the experimental currency to avoid expressing value/cost using any particular country’s currency. This has facilitated market demonstrations using the trading software with a variety of industry/government officials in New Zealand, Australia, Argentina and Spain.

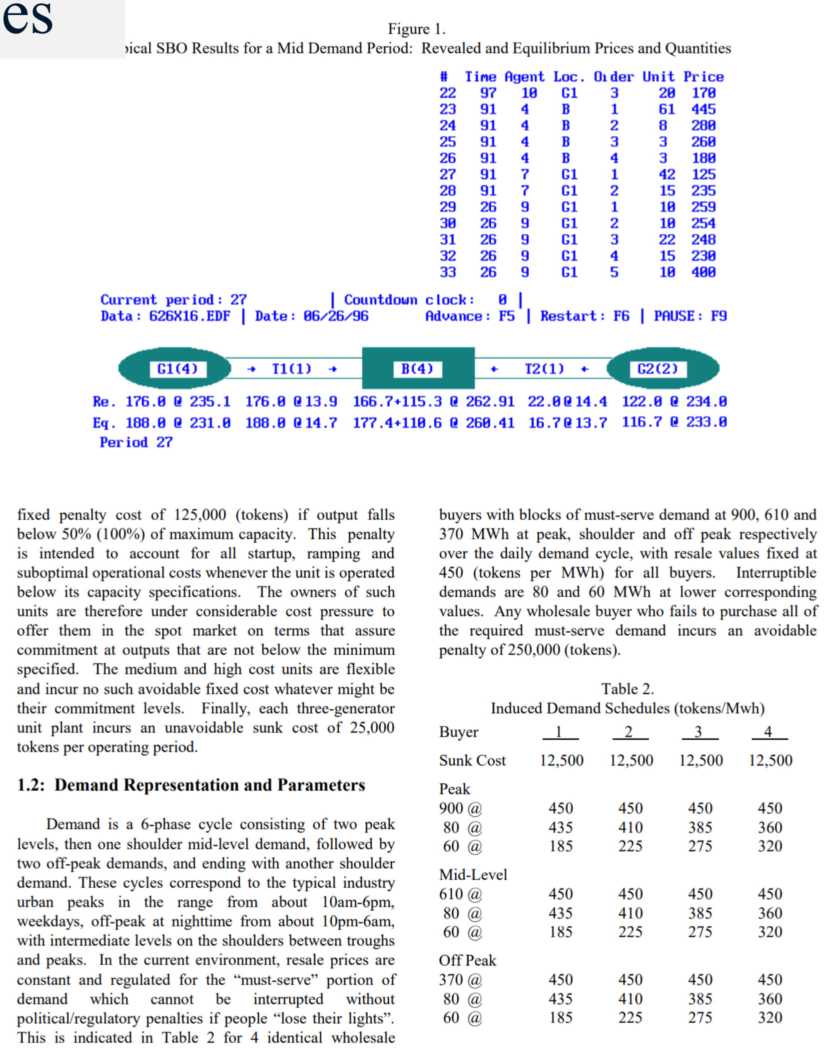


Figure 1.
ical SBO Results for a Mid Demand Period: Revealed and Equilibrium Prices and Quantities

```
                                      #  Time Agent Loc. Order Unit Price
                                      22  97   10   G1    3     20  170
                                      23  91   4    B     1     61  445
                                      24  91   4    B     2     8   280
                                      25  91   4    B     3     3   260
                                      26  91   4    B     4     3   180
                                      27  91   7    G1    1     42  125
                                      28  91   7    G1    2     15  235
                                      29  26   9    G1    1     10  259
                                      30  26   9    G1    2     10  254
                                      31  26   9    G1    3     22  248
                                      32  26   9    G1    4     15  230
                                      33  26   9    G1    5     10  400

 Current period: 27          | Countdown clock:   0 |
 Data: 626X16.EDF | Date: 06/26/96      Advance: F5 | Restart: F6 | PAUSE: F9

      G1(4)     →  T1(1)  →      B(4)        ←  T2(1)  ←     G2(2)

Re. 176.0 @ 235.1  176.0 @13.9  166.7+115.3 @ 262.91  22.0@14.4  122.0 @ 234.0
Eq. 188.0 @ 231.0  188.0 @14.7  177.4+110.6 @ 260.41  16.7@13.7  116.7 @ 233.0
 Period 27
```

fixed penalty cost of 125,000 (tokens) if output falls below 50% (100%) of maximum capacity. This penalty is intended to account for all startup, ramping and suboptimal operational costs whenever the unit is operated below its capacity specifications. The owners of such units are therefore under considerable cost pressure to offer them in the spot market on terms that assure commitment at outputs that are not below the minimum specified. The medium and high cost units are flexible and incur no such avoidable fixed cost whatever might be their commitment levels. Finally, each three-generator unit plant incurs an unavoidable sunk cost of 25,000 tokens per operating period.

1.2: Demand Representation and Parameters

Demand is a 6-phase cycle consisting of two peak levels, then one shoulder mid-level demand, followed by two off-peak demands, and ending with another shoulder demand. These cycles correspond to the typical industry urban peaks in the range from about 10am-6pm, weekdays, off-peak at nighttime from about 10pm-6am, with intermediate levels on the shoulders between troughs and peaks. In the current environment, resale prices are constant and regulated for the “must-serve” portion of demand which cannot be interrupted without political/regulatory penalties if people “lose their lights”. This is indicated in Table 2 for 4 identical wholesale buyers with blocks of must-serve demand at 900, 610 and 370 MWh at peak, shoulder and off peak respectively over the daily demand cycle, with resale values fixed at 450 (tokens per MWh) for all buyers. Interruptible demands are 80 and 60 MWh at lower corresponding values. Any wholesale buyer who fails to purchase all of the required must-serve demand incurs an avoidable penalty of 250,000 (tokens).

Table 2.
Induced Demand Schedules (tokens/Mwh)

Buyer	1	2	3	4
Sunk Cost	12,500	12,500	12,500	12,500
Peak				
900 @	450	450	450	450
80 @	435	410	385	360
60 @	185	225	275	320
Mid-Level				
610 @	450	450	450	450
80 @	435	410	385	360
60 @	185	225	275	320
Off Peak				
370 @	450	450	450	450
80 @	435	410	385	360
60 @	185	225	275	320

The California Politicians Ran Ahead of the Research, Resulting in Their 2000-2001 Crisis

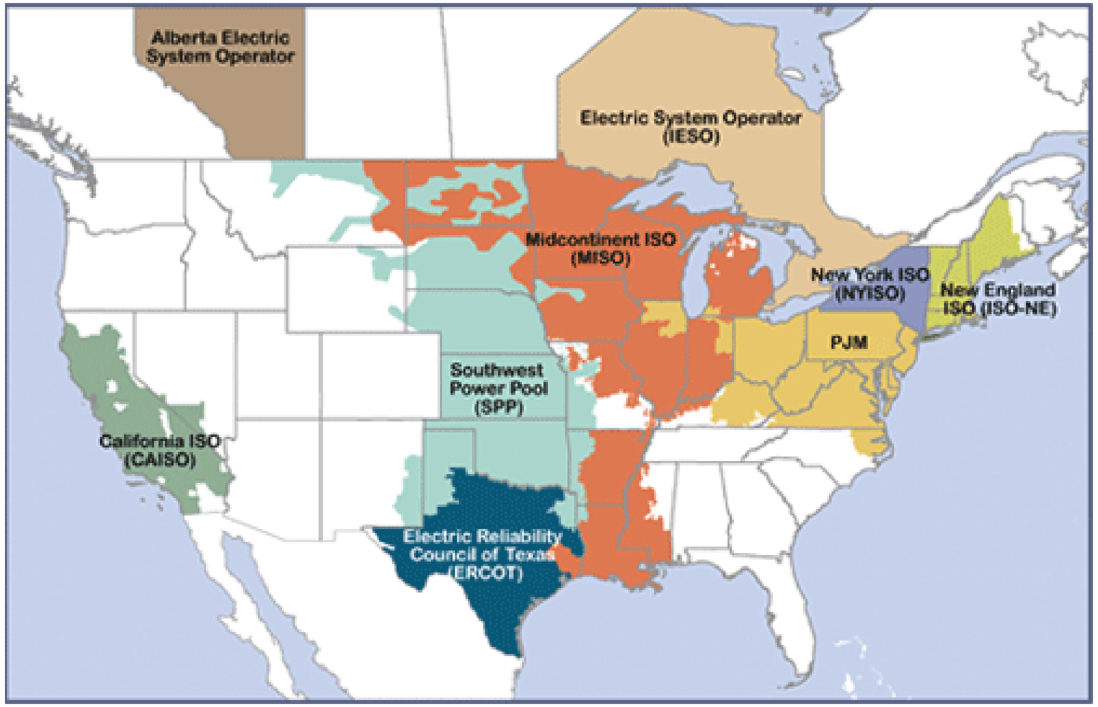
- In 1996 California decided to create an electricity market even though the risks in doing this were not fully known.
- During 2000 their wholesale electricity prices jumped by 800% due in part to market manipulation.

MARK FIORE, San Francisco

Y2.1K

Electricity Markets Today in North America

- Starting in about 1995 electricity markets gradually started to develop, both in the US and elsewhere.
- In North America more than 60% of the load is supplied via wholesale electricity markets; markets differ but they all have certain common features.



  - The terms regional transmission organizations (RTOs) and independent system operators (ISOs) are used (RTOs are more functionality and most are actually RTOs
- Image source: www.ferc.gov/industries-data/electric/power-sales-and-markets/rtos-and-isos

Electricity Markets Common Features

- Day ahead market – this is needed because time is required to make decisions about committing generators.
  - Generation owners submit offers for how much generation they can supply and at what price; accepted offers are binding
- Real-time energy market – needed because day ahead forecasts are never perfect, and unexpected events can occur.
- Co-optimization with other "ancillary services" such as reserves.

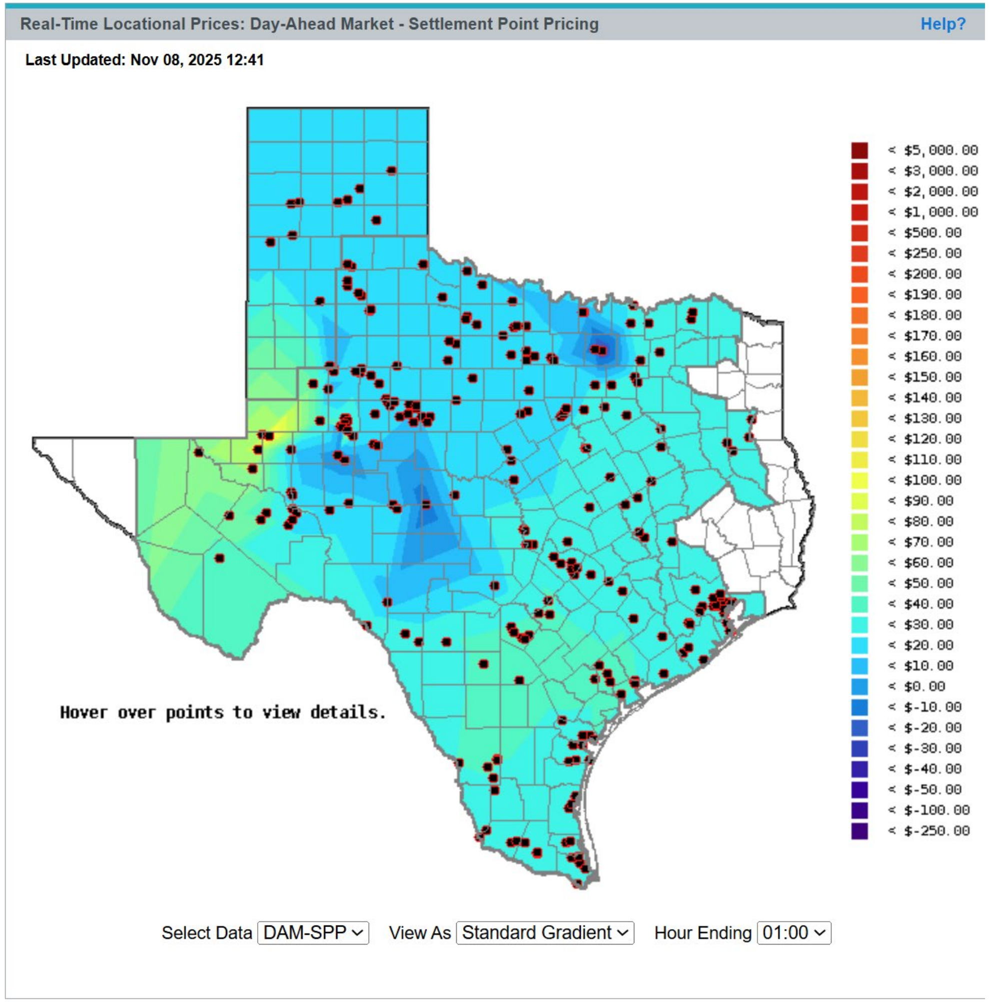
The source for much of this material "Analytic Research Foundations for the Next-Generation Electric Grid" (Chapter 2), The National Academies Press, 2016 (free download available)

Electricity Markets Common Features

- Pricing is done using locational marginal prices, determined by an SCOPF.
  - Most markets include a marginal losses component
- LMP markets are designed to send transparent price signals so people can make short and long-term decisions.
  - Generators are free to offer their electricity at whatever price they desire; they do not have to reveal their "true" costs
  - Most of the times markets work as planned (competitive prices)
  - During times of shortages (scarcity) there are limits on LMPs; ERCOT's had been $9000/MWh prior to Uri; now it is $5000/MWh
  - Markets are run by independent system operators (ISOs)

ERCOT November 8, 2025 LMPs (13:15 CST)

Day Ahead Market



Real-Time Locational Prices: Day-Ahead Market - Settlement Point Pricing

Last Updated: Nov 08, 2025 12:41

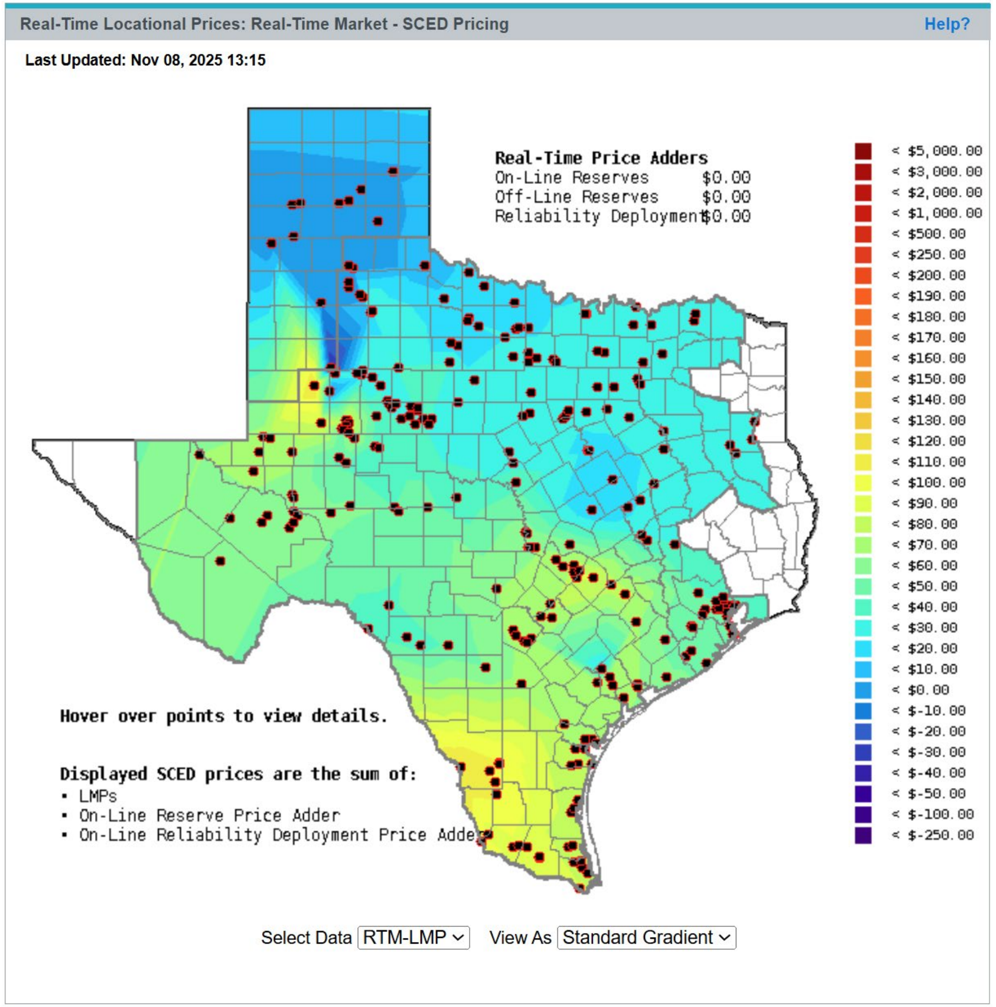
Hover over points to view details.

Select Data DAM-SPP

View As Standard Gradient

Hour Ending 01:00

Real-Time Market



Real-Time Locational Prices: Real-Time Market - SCED Pricing

Last Updated: Nov 08, 2025 13:15

Real-Time Price Adders
On-Line Reserves $0.00
Off-Line Reserves $0.00
Reliability Deployment $0.00

Hover over points to view details.

Displayed SCED prices are the sum of:
- LMPs
- On-Line Reserve Price Adder
- On-Line Reliability Deployment Price Adder

Select Data RTM-LMP

View As Standard Gradient